Catastrophic risk and thresholds in resource economics

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Purpose of class

- Teach some advanced methods in optimal control theory
- Familiarize students with some applications of these methods to natural resource management.
- A very applied course. No theorems; just methods!

Prerequisites

- A decent understanding of ordinary differential equations.
- A decent understanding of deterministic optimal control theory.
- E.g. $\text{Max}_c \int_0^T U(c,x)e^{-rt}dt$ subject to $x(0) = x0$ and $dx/dt=f(x,c)$. Here $0 < T \leq \infty$.

Deterministic Threshold Problems Irreversible case.

- A threshold is here defined to be some curve in state space such that crossing this curve leads to a discrete jump in state-variables.
- Preliminary observation; if the location of the threshold is known one can choose to cross the threshold or not cross the threshold.
- An irreversible threshold effect can be modeled as a scrap value problem with endogenous time horizon

Mathematical description

- From Seierstad and Sydsæter Theorem 3 and Note 2, pp 182.
- Let x be of dimension n
- Let the threshold be given by $R(x) = 0$
- Let τ be the first point in time when the threshold is reached. We assume that is $R(x(\tau)) = 0$.
- Let the threshold effect be $x(\tau^+)-x(\tau) = g(x(\tau))$
  - $x(\tau^+) = x(\tau^-) = \lim_{t\to\tau} x(t)$ from below
  - $x(\tau^+) = \lim_{t\to\tau} x(t)$ from above

Formulating the Optimization Problem(s) - 1 Going over the threshold

- $\text{Max}_{c,\tau} \int_0^\tau U(c,x)e^{-rt}dt + S(x(\tau) + g(x(\tau)))\, e^{-r\tau}$ subject to $x(0) = x0$, $dx/dt = f(x,c)$ and $R(x(\tau)) = 0$.

Here we should think of S $S(x(\tau) + g(x(\tau)))$ as a value function. In fact $S(x)$ is given by

$\text{Max}_{c,\tau} \int_\tau^\infty U(c,y)e^{-rt}dt$ subject to $y(0) = x$ and $dy/dt = f(y,c)$.

Solve the problem recursively

- From $Max_c \int_{\tau}^{\infty} U(c,y)e^{-rt}dt$ subject to $y(0) = x$ and $dy/dt=f(y,c)$ we get the shadow price $\lambda(x|\tau)$.

Optimality conditions (Present value)

- u maximizes the Hamiltonian
- $d\lambda/dt = -\partial H/\partial x$
- $\lambda(\tau) = \lambda(x(\tau)|\tau)(1+g'(x)) + \gamma\partial R/\partial x$

Note that all these may be vectors

- If τ lies in $(0, \infty)$, then
- $H + \partial(S(x(\tau)+ g(x(\tau)))\, e^{-r\tau})/\partial t = 0$

Optimality conditions (Present value)

- u maximizes the Hamiltonian
- $d\lambda/dt = -\partial H/\partial x$
- $\lambda(\tau) = \lambda(x(\tau)|\tau)(1+g'(x)) + \gamma\partial R/\partial x$

Note that all these may be vectors

- If τ lies in $(0, \infty)$, then
- $H + \partial(S(x(\tau)+ g(x(\tau)))\, e^{-r\tau})/\partial t =$

$H - r(S(x(\tau)+ g(x(\tau)))\, e^{-r\tau}) = 0.$

Formulating the Optimization Problem(s) - 2 Going over the threshold

- $\text{Max}_{c,\tau} \int_0^{\tau} U(c,x)e^{-rt}dt + S(x(\tau)))e^{-r\tau}$ subject to $x(0) = x0$, $dx/dt=f(x,c)$ and $R(x(\tau)) = 0$.

Again we should think of $S(x(\tau))$ as a value function. In fact $S(x)$ is given by

$\text{Max}_{c,\tau} \int_{\tau}^{\infty} U(c,y)e^{-rt}dt$ subject to $y(0) = x$, $dy/dt=f(y,c)$ and $R(x(t)) = 0$.

Solve the problem recursively

- From $Max_c \int_{\tau}^{\infty} U(c,y)e^{-rt}dt$ subject to $y(0) = x$, $dy/dt=f(y,c)$, and $R(x(t)) = 0$. We get the shadow price $\lambda^*(x|\tau)$.

- Note: Here I have excluded the possibility that $R(x(t)) \neq 0$ for some $t > \tau$. It is however perfectly possible that we may "turn away" from the threshold at some point in time. In particular if we are studying a finite horizon problem

Optimality conditions (Present value)

- u maximizes the Hamiltonian
- $d\lambda/dt = -\partial H/\partial x$
- $\lambda(\tau) = \lambda^*(x(\tau)|\tau) + \gamma\partial R/\partial x$

Note that all these may be vectors

- If τ lies in $(0, \infty)$, then
- $H + \partial(S(x(\tau)))\, e^{-r\tau})/\partial t = 0$

A simple threshold model (Nævdal 2003)

- Let $dx/dt = u - \delta x$.
- Let the threshold be x'.
- Let B(u) be maximised at B(u*) for u* > 0.
- Let the let the utility function be A + B(u) if the threshold is not crossed and let the benefit function be B(u) if it is crossed.

Avoid Crossing the threshold?

- Let T_e be the time at which the threshold is crossed.
- If we do not cross the threshold, then after getting to $x(T_e) = x'$ we must freeze u. Thereafter there is no need to reduce u so $u = \delta x'$ for all $t > T_e$.
- Then $S_e(x(T_e))e^{-rTe} = e^{-rTe} \int_{Te}^{\infty} (A+B(u^*))e^{-rt}dt$

Crossing the threshold?

- Let T_a be the time at which the threshold is crossed.
- If we cross the threshold, then we accept the damage. Thereafter there is no need to reduce u so u = u* for all $t > T_a$.
- Then $S_a(x(T_a))e^{-rT_a} = e^{-rT_a} \int_{T_a}^{\infty} B(u^*)e^{-rt}dt$

Optimality conditions prior to hitting the threshold

- Hamiltonian is the same for both problems
- $H = (A + B(u))e^{-rt} + p(u - \delta u)$

Gives that $B'(u_i)e^{-rt} + p_i = 0$

$dp_i/dt = \delta p_i \rightarrow p_i(t) = C_i e^{-\delta t}$

$x(T_i) = x'$ $i = a,e$

The shadow prices after hitting or crossing the threshold are both zero.

u is a decreasing function of time. Pollute early and then reduce

Note

- Both are problems with variable final time and the requirement that $x_i(T) = x'$. We need condition to find T_i.

Going over the threshold

- Condition for optimal T_a.

$$
\begin{aligned}
H &= (A + B(u_a(T_a)))e^{-rT_a} + p_a(u_a(T_a) - \delta\bar{x}) \\
&= -\frac{dJ_a}{dT_a} = (A + B(u^*))e^{-rT_A}
\end{aligned}
$$

$$
A + B(u_a(T_a)) - B(u^*) = B'(u_a(T_a))(u_a(T_a) - \delta\bar{x})
$$

- We can prove that u_a is discontinuous at T_a.

Optimal paths of u_a and x_a

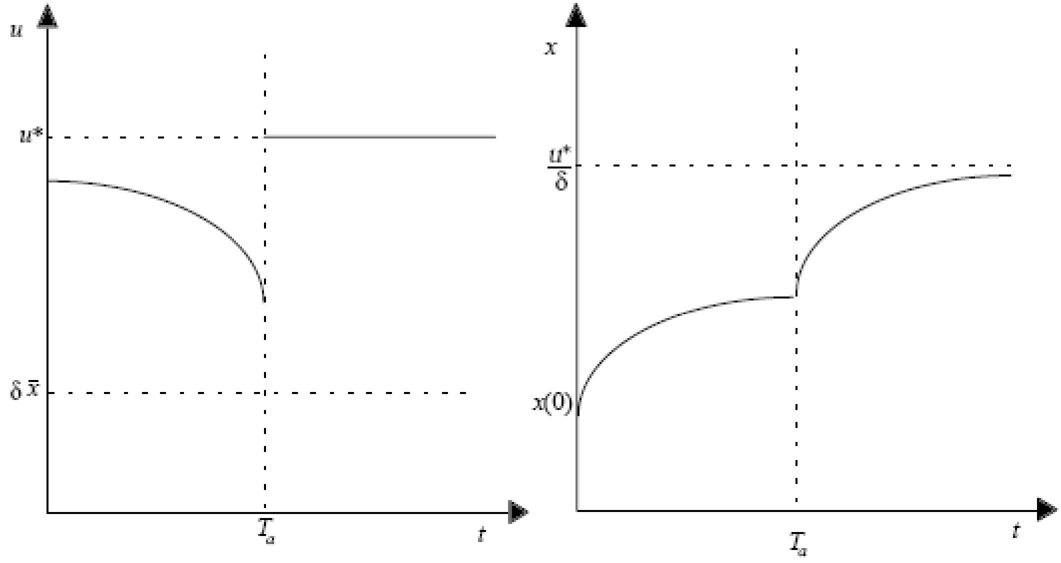


FIGURE 1. "Into the abyss".

Staying on the edge

- Condition for optimal T_e.

$$H = (A + B(u_e(T_e)))e^{-rT_e} + p_e(u_e(T_e) - \delta\bar{x})$$
$$= -\frac{dJ_e}{dT_e} = (A + B(\delta\bar{x}))e^{-rT_e}$$

- This condition can be used to prove that $u_e(t)$ is continuous at T_e.

Optimal paths of u_e and x_e

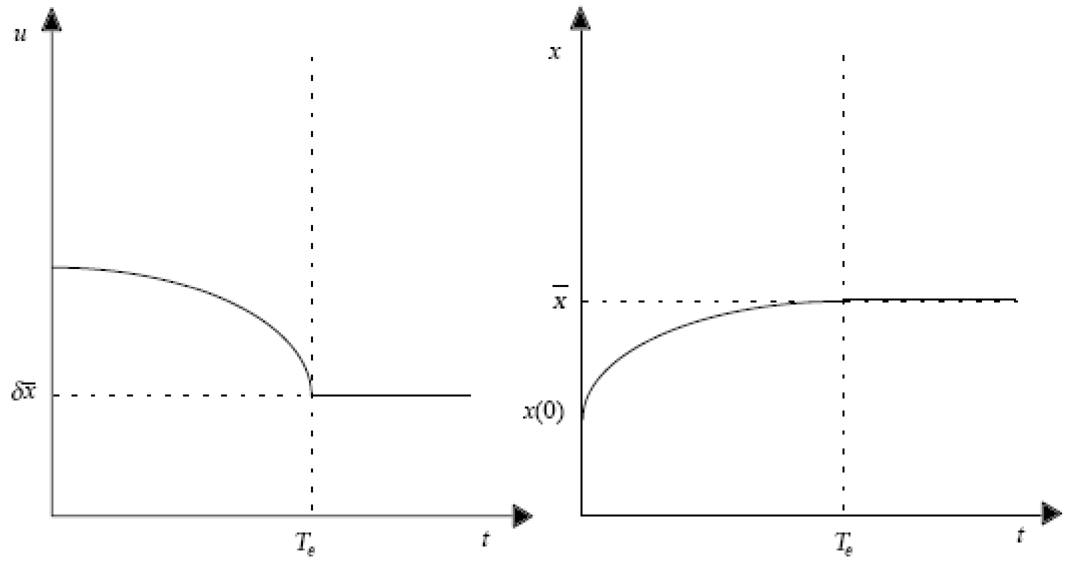


FIGURE 2. "Onto the edge".

Important to note

- There are kinks (u_e) or jumps (u_a) in the optimal paths of the control.
- The previous literature on deterministic thresholds had overlooked:

1. That a threshold could either be crossed or observed
2. That optimal controls may be discontinuous at the time the threshold is crossed. (In some models this can happen when the threshold is reached even if it is not crossed.)

Comparing Scenarios

Proposition.

I) $T_a < T_e$.

II) $u_a(t) > u_e(t)$ *for all* t.

Proof. $u_a(T_a)$ is larger than $u_e(T_e)$ since $u_a(T_a) - \delta\bar{x} > 0 = u_e(T_e) - \delta\bar{x}$. Since u_a and u_e only differ by the magnitude of a constant C, this implies that u_a is larger than u_e for all t in the interval $[0, \min(T_e, T_a)]$ which again implies that $T_a < T_e$ which proves I). But if I) holds, it is trivial that II) holds for the interval (T_a, ∞), which concludes the proof. □

I said no proofs, but this one is rather instructive and a good example of how simple proofs can give interesting results

Incorporating uncertainty into optimal control

- Most processes are subject to uncertainty
- One class is Brownian motion. I will not talk about that at all.
- Catastrophic events.
  - Tsunamis
  - Floods
  - Car breakdown

Categories of catastrophic uncertainty

- Endogenous vs. exogenous risk
  - Some processes we can (partially) control. This is endogenous risk.
  - Some processes are truly beyond our abilities. Exogenous risk.
- State-space distributed risk vs time-distributed risk.
  - State space distributed risk requires movement through state space. (thresholds)
  - Time distributed risk depends on location in state- space. For a fixed location in state-space, uncertainty is invariant with respect to time. This will perhaps become clearer. (I hardly understand it myself)

Preliminaries – Poisson processes in continuous time

- We are here concerned with events that occur at random points in time. In order to deal with these problems they must have some kind of distribution that we actually know.
- We shall refer to the point in time when the event occurs as τ (tau) and it is distributed over a subset of the positive real numbers $[0, \beta)$ where $0 < \beta \leq \infty$.
- The pdf is $g(t)$ and the cdf is $G(t) = \int_0^t g(s)ds$.

Poisson processes and conditional updating.

- Our process starts t = 0. We make it to t* >0. What is the distribution of τ conditional on us having made it to t*?

Answer:

$$g\left(t \mid t > t^*\right) = \frac{g(t)}{\int_{t^*}^{\beta} g(s)\,ds}$$

We need optimality conditions that reflect updating of the distribution as long as τ does not happen.

The hazard rate - A very useful concept

- We ask the question; What if we have made it to time t? What is the probability that τ occurs in the time interval (t, t + Δt)? ∈
- If dt is small, then that probability is roughly Pr(τ (t, t + Δt))=g(t|τ>t)×dt.
- g(t|τ>1) is of course:

$$g\left(t \mid t > t_1\right) = \frac{g(t)}{\int_t^{\beta} g(s)\,ds}$$

Hazard rate μ(t)– Formal definition

$$\mu(t) = \lim_{\Delta t \to 0^+} \frac{\Pr\left(\tau \in (t, t+\Delta t) \mid \tau > t\right)}{\Delta t}$$

$$= \frac{g(t)}{\int_t^b g(s)\,ds} = \frac{g(t)}{1 - \int_0^t g(s)\,ds} = \frac{g(t)}{1 - G(t)}$$

Example - the exponential distribution

- Let μ^0 be a positive constant number.
- $g(t) = \mu^0 e^{-\mu^0 t}$. $\rightarrow$ $G(t) = 1 - e^{-\mu^0 t}$.
- Then $\mu(t) = \mu^0$. We have a constant hazard rate. This is only true for the exponential distribution.
- Interesting fact: $E(\tau - t) = {}_t\int^{\infty} s\mu e^{-\mu s} ds = 1/\mu$. The expected conditional waiting time is constant
- We can in principle calculate hazard functions for any g(t). They are frequently messy.

We have the hazard rate – What is the pdf?

- A very useful property. Let $\mu(t)$ be any function that is:
  - continuous and positive for all t
  - has the property that $_0\int^\beta \mu(t)dt=\infty$.
- Then we can construct a cdf by solving the following differential equation.

$$\mu(t) = y'(t)/(1 - y(t)), \quad y(0) = 0$$

Finding a pdf

$\mu(t) = y'(t)/(1 - y(t)), \quad y(0) = 0$

implies that $y(t) = 1 - C\times\text{Exp}(-\int^t \mu(s)ds)$

Here C is a constant determined by the boundary condition. For many choices of hazard rate this constant will be equal to 1.

Having found y(t), the cdf, we can find the pdf

$y'(t) = C\times\mu(t)\times\text{Exp}(-\int^t \mu(s)ds)$

Example – linear hazard rate

- Assume that $\mu(t) = \mu^0 t$ defined for t $[0, \infty)$. Then $\mathrm{Exp}(-\int^t \mu(s)ds) = ½\mu^0 t^2$. Therefore

$y(t) = 1 – C\times \mathrm{Exp}(-½\mu^0 t^2)$. $y(\infty) = 1$ implies that C = 1, so the cdf is $1 – \mathrm{Exp}(-½\mu^0 t^2)$ and the pdf is $y'(t) = \mu^0 t\times \mathrm{Exp}(-½\mu^0 t^2)$.

Hazard rate continued

- The previous results also hold if the hazard rate can be written $\mu(t) = \varphi(x(t))$ (or $\varphi(t,x(t))$ for that matter) where *x*(t) is any function of *t*.
- This holds as long as:
  - $\varphi(x(t))$ only takes positive values
  - The integral $\int \varphi(x)dx$ does not converge
  - The integral $\int \varphi(x(t))dt$ may converge (and will usually do so)
- Thus we can work with very general density pdfs over time over time: $\varphi(x(t))\mathrm{Exp}(- \int^t \varphi(x(s))ds)$

Nice ting to know about hazard rates

- If we have two stochastic events τ_1 and τ_2, with hazard rates λ_1 and λ_2, then….
- The event $\tau = \text{Min}(\tau_1, \tau_2)$ has a hazard rate

$$\lambda = \lambda_1 + \lambda_2.$$

Practical as it enables us to treat two stochastic processes as one while keeping consequences separate!

Exogenous vs endogenous uncertainty

- Slightly confusing literature. Here the difference is as follows.
- If $\varphi(x(t))$is the hazard rate and $x(t)$ is determined by a controllable differential equation, then the stochastic process is endogenous.
- If not, the process is exogenous
- This is not clear cut. Hurricanes may be (in part) endogenous to US policymakers but exogenous to the good citizens of New Orleans

Controlling exogenous catastrophic uncertainty

- Basic problem: Nature (or somebody we can't affect) triggers a catastrophic event.
- We can not control the probability of the event occurring, but we can control:
  - preparedness (what to do before the event)
  - consequences (how to act after the event)
- No strict boundary between preparedness and consequence management.

The mathematical formulation

- In infinite time:

Max $E\{\int^{\infty}(U(x,u)e^{-rt})dt + h(x(\tau^-))$

subject to $dx/dt=f(x,u)$ for all $t \neq \tau$

$x(\tau^+)-x(\tau^-) = g(x(\tau^-))$

τ distributed $\mu e^{-\mu t}$ over $[0, \infty)$

$g(x(\tau^-))$ is the physical description of the event

$h(x(\tau^-))$ is the instantaneous value of the shock.

$x(\tau^-) = \lim_{t\rightarrow\tau} x(t)$ from below

$x(\tau^+) = \lim_{t\rightarrow\tau} x(t)$ from above

Extreme example

- A tsunami in a community
  - We have utility before the disaster. Depends on consumption and stock of capital.
  - The instantaneous cost of the disaster, depends on stochasticity and the stock of preparedness capital.
  - Utility after the disaster, depends on consumption and the stock of capital that survived the disaster

Solving the problem a recursive algorithm

- First find the optimal policy after the disaster.
- Find optimal policy before the disaster.
- Sounds pretty simple…

Post-disaster control

- There is no stochasticity. Simply solve the following problem:

$J(t,x|\tau = t)e^{rt} = \max_u \; e^{rt} \int_t^{\infty} U(u,y)e^{-rs}ds$ subject to $y(0) = x$ and $dy/dt=f(y,u)$.

- Here t and x are arbitrary. The indicate the possible state(s) of the world after a disaster.
- The notation $J(t,x|\tau = t)$ indicates that J is evaluated conditional on τ happening at time t.

Pre-disaster control

- From the post disaster control we will need $J'_x(t,x|\tau = t)e^{rt} = \lambda(x|\tau)$. This is the shadow price of *x* at the instant after the catastrophic event occurs.
- In this problem, that is really all we need from the post-disaster program.

Pre-disaster program- Necessary conditions Hazard rate depends on t.

- Define the Hamiltonian:

$$H = U(u,x) + \lambda f(u,x) + \mu(t)(J(x+g(x)|\tau) - J(x) + h(x))$$

Utility today

Marginal value of
X × increase in x

Hazard rate

Change in utility if
Catastrophic event occurs

Optimality Conditions

- Apply the maximum principle to the Hamiltonian.
- u =argmax H
- $d\lambda/dt = r\lambda - \partial H/\partial x$
- $dx/dt=f(x,u)$

- Transversality condition as in deterministic models.

The differential equation for λ

- $d\lambda/dt = r\lambda - \partial H/\partial x$

  $= r\lambda - \partial U/\partial x - \lambda\partial f/\partial x$

  $- \mu((\partial J(x+g(x)|\tau)/\partial x - \partial J(x)/\partial x) + h'(x))$

But $\partial J(x+g(x)|\tau)/\partial x = \lambda(x|\tau)(1 + g'(x))$ and $\partial J(x)/\partial x = \lambda$, so:

$d\lambda/dt = r\lambda - \partial U/\partial x - \lambda\partial f/\partial x + \mu(\lambda - \lambda(x|\tau)(I_n + g'(x)) - h'(x))$

Note: I_n is the identiy matrix where n is the number of elements in x(t)

Example

Let $U(x,u) = -ax - ½(u^0 – u)^2$ and $f(x,u) + u – \delta x$

Let τ be exponentially distributed with hazard rate μ. Let $x(\tau^+)-x(\tau^-) = \beta x(\tau^-)$. $h(x) = 0$.

Could be a model of pollution where there is a possibility that a shock increases the stock of pollutants.

Optimality conditions after event

$H = -ax - \frac{1}{2}(u^0 - u)^2 + \lambda(u - \delta x)$

1. The value of u that maximizes the Hamiltonian is given by $u = u^0 + \lambda$ for $u^0 + \lambda > 0$. Else $u = 0$.
2. $d\lambda/dt = r\lambda + a + \delta\lambda$
3. $dx/dt = u^0 + \lambda - \delta x$

$d\lambda/dt = r\lambda + a + \delta\lambda \rightarrow \lambda(t|\tau) = -a/(r + \delta)$ for all t. We make note of $\lambda(t|\tau)$ and proceed to the pre-event control

Pre event conditions

1. The value of u that maximizes the Hamiltonian is given by $u = u^0 + \frac{1}{2}\lambda$ for $u^0 + \frac{1}{2}\lambda > 0$. Else $u = 0$. Same as before!
2. $dx/dt = u^0 + \frac{1}{2}\lambda - \delta x$. Same as before!
3. $d\lambda/dt = r\lambda + a + \delta\lambda + \mu(\lambda - (-a(1+\beta)/(r + \delta)))$

Here is the difference in the conditions. Obviously this implies that u and x will be affected by the risk

We can solve for λ

- Solution is:

$$\lambda(t) = -\frac{a(r+\delta+\mu(1+\beta))}{(r+\delta)(r+\delta+\mu)} + Ke^{(r+\delta+\mu)t}$$

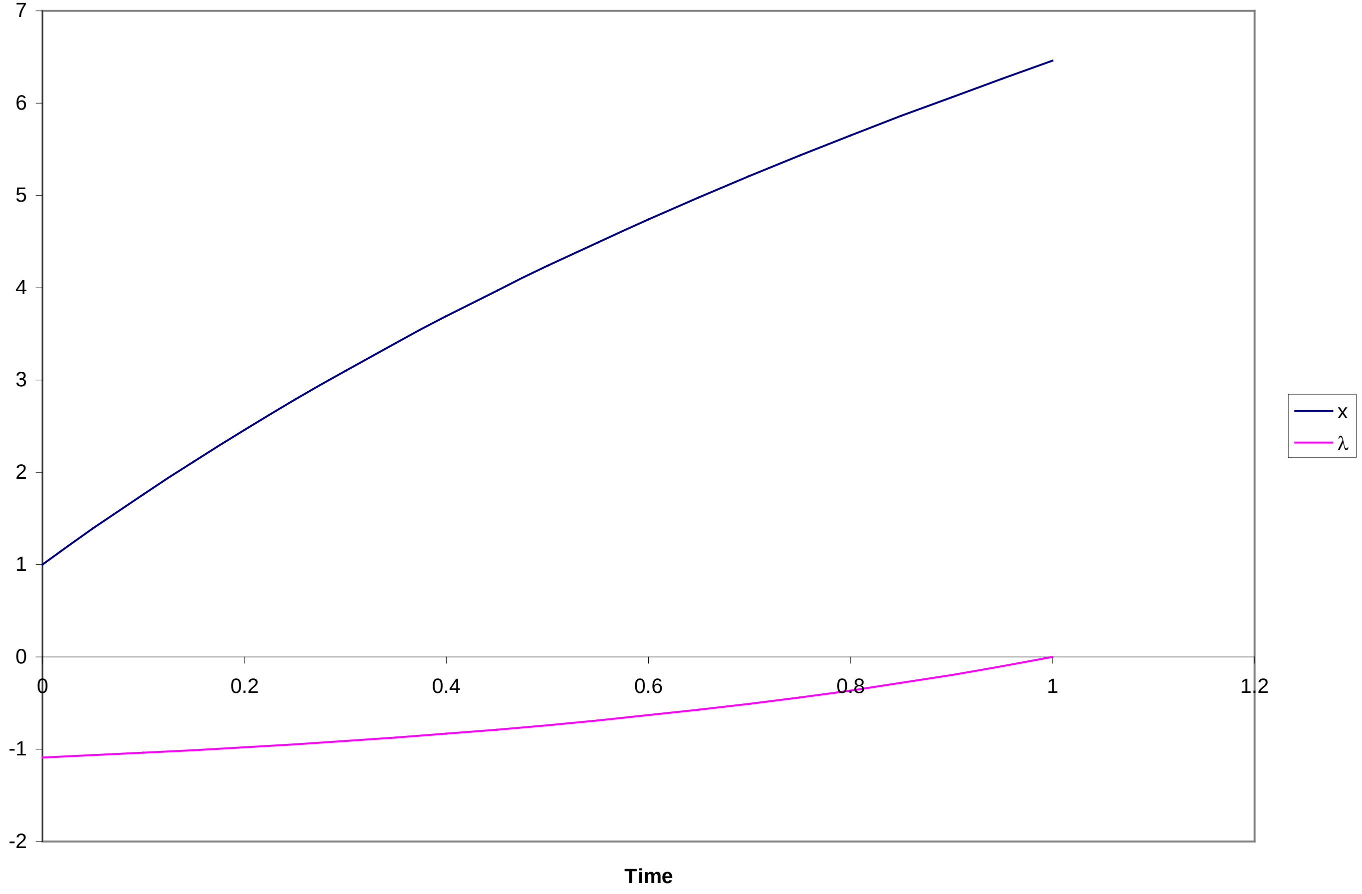
- Here K is a constant that must be determine by transversality conditions. We just note that $K \neq 0$ implies no convergence to steady state so we just set $K = 0$. Note that if $\beta = 0$, then pre-event solution is same as post-event

The we have that….

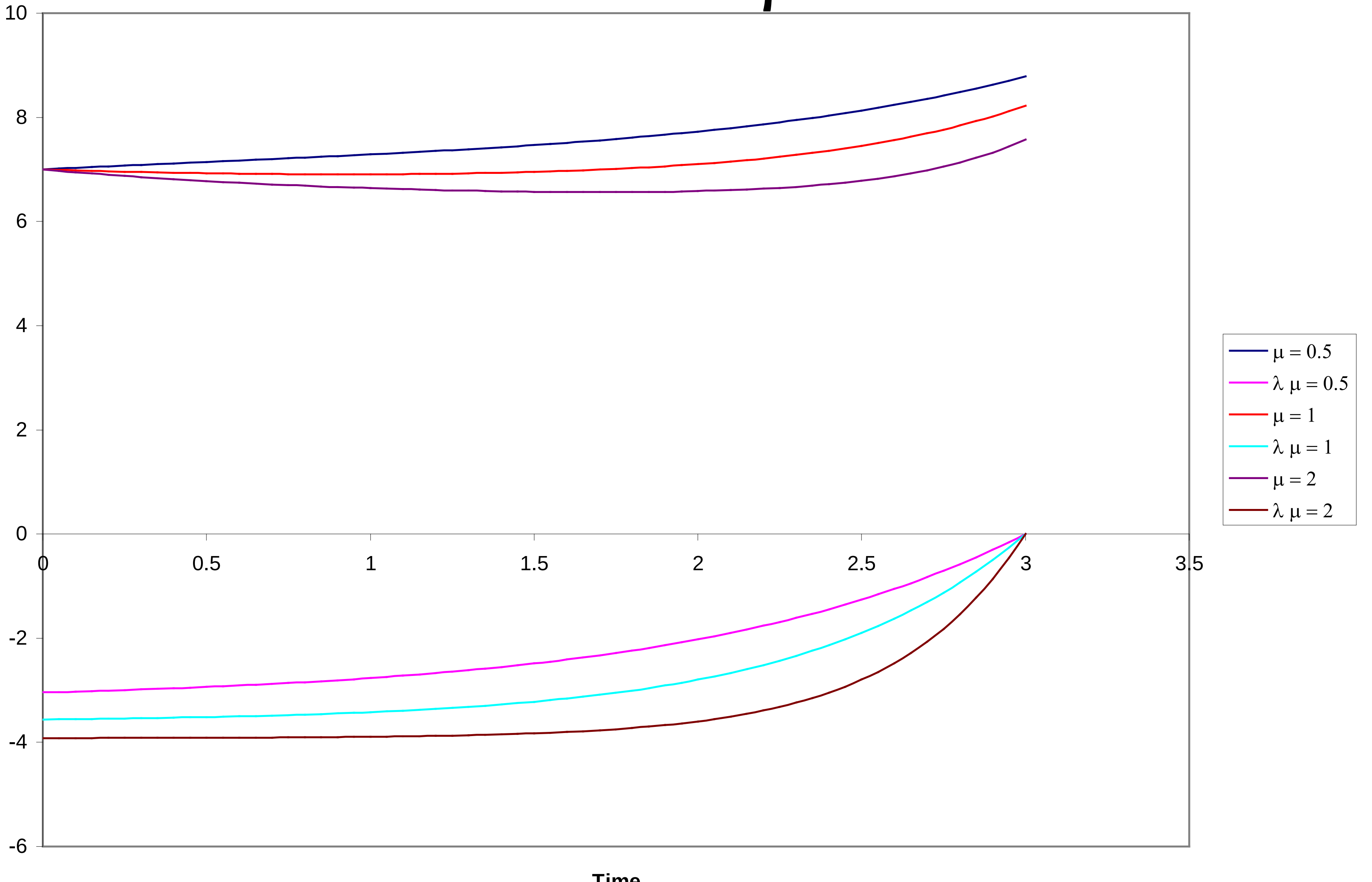
- The optimal value of u(t) is found by inserting $\lambda(t)$ into $u^0 + \frac{1}{2}\lambda$.
- The optimal value of x(t) is fund by integrating $dx/dt = u^0 + \frac{1}{2}\lambda - \delta x$

- I will not do this as the resulting expression is a bugger.
- Numeric solutions are straightforward to find

Result

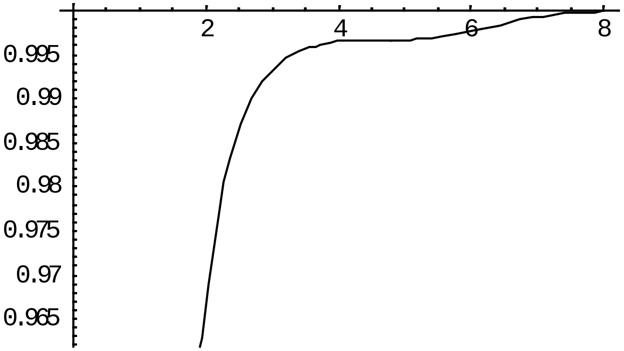


The effect of uncertainty – Let us increase μ



Weird hazard rates

- The setup is good enough to include time dependent hazard rates. That is, the hazard rate depends on time.
- Let us consider a hazard rate a×(Sin(t) + 1). This implies a cdf given by 1 – exp(-1-t+Cos(t)). Cdf looks like this:



Pre-disaster program- Necessary conditions Hazard rate depends on t.

- Define the Hamiltonian:

$$H = U(u,x) + \lambda f(u,x) + \mu(t)(J(x+g(x)|\tau) - J(x) + h(x))$$

Utility today

Marginal value of
X × increase in x

Hazard rate

Change in utility if
Catastrophic event occurs

Optimality Conditions

- Apply the maximum principle to the Hamiltonian.
- u =argmax H
- $d\lambda/dt = r\lambda - \partial H/\partial x$
- $dx/dt=f(x,u)$

NO CHANGE!

- Transversality condition as in deterministic models.

The differential equation for λ

- $d\lambda/dt = r\lambda - \partial H/\partial x$

  $= r\lambda - \partial U/\partial x - \lambda\partial f/\partial x$

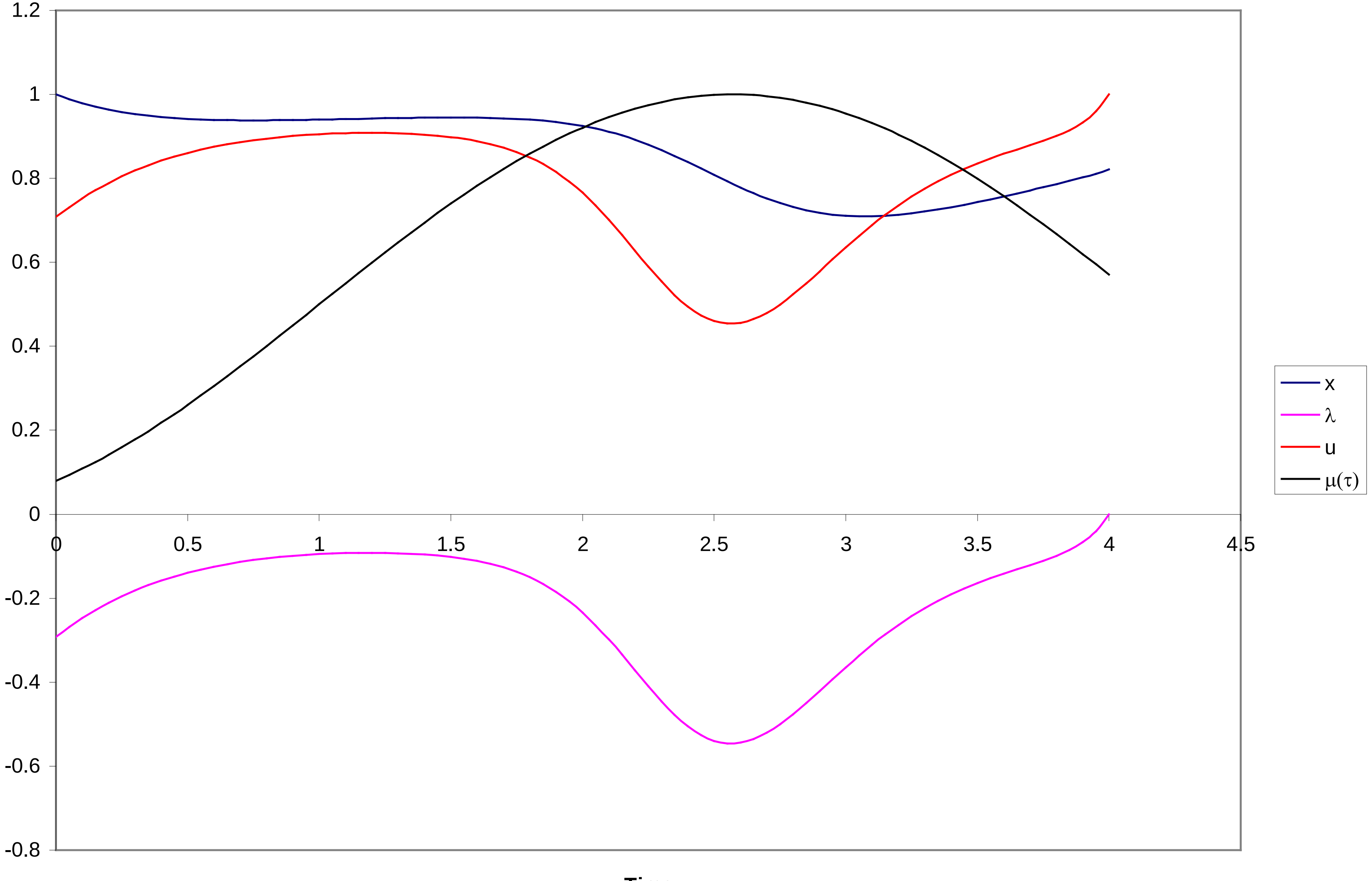
  $- \mu(t)((\partial J(x+g(x)|\tau)/\partial x - \partial J(x)/\partial x)+h'(x))$

But $\partial J(x+g(x)|\tau)/\partial x = \lambda(x|\tau)(1 + g'(x))$ and $\partial J(x)/\partial x = \lambda$, so:

$d\lambda/dt = r\lambda - \partial U/\partial x - \lambda\partial f/\partial x + \mu(t)(\lambda - \lambda(x|\tau)(I_n + g'(x)) - h'(x))$

Note: I_n is the identiy matrix where n is the number of elements in x(t)

Results



Endogenous risk – Time distributed

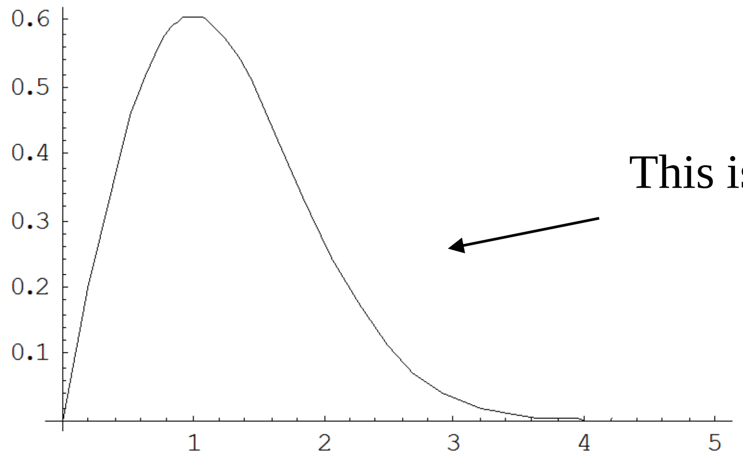
- Recall that we know face a controllable hazard rate.
- The hazard rate depends on a state variable

Endogenous time distributed risk

- Here the hazard rate will depend on the state variable.

  $\mu = \mu(x)$.

Example $\mu(x) = \mu^0 x$



This is the pdf

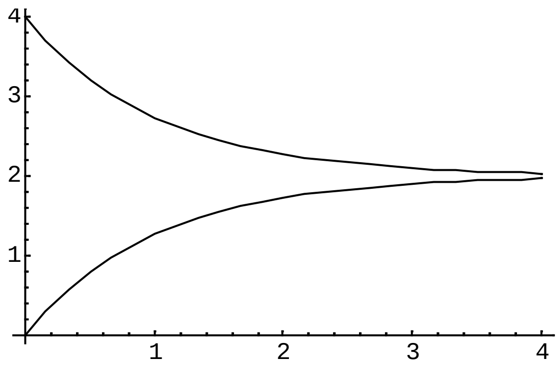
- Remember that x is a function of t because it is determined by an differential equation

Endogenous risk continued

- Say that x(t) is driven by $dx/dt = -ax + b$, $x(0) = X$. Then $x(t) = e^{-at}(X - b/a) + b/a$.
- The hazard rate is then transformed from a function of x to a function of time.

$$\mu = \mu 0 \times (e^{-at}(X - b/a) + b/a)$$

Illustration for different starting points of x(0)



Controlling Endogenous Uncertainty

- The general problem

$$J(0, x(0)) = \max_{u \in U} E\left(\int_0^T f(x, u) e^{-rt} \, dt \right),$$

$$\text{s.t}: \quad u \subseteq \mathbb{R}^m, \quad x(0) \subseteq \mathbb{R}^n, \quad \dot{x} = g(x, u),$$

$$\tau \sim \lambda(x(t)) e^{-\int_0^t \lambda(x(\sigma)) \, d\sigma} \ \text{over} [0, \infty),$$

$$x(\tau^+) - x(\tau^-) = q(x(\tau^-)).$$

Recursivity - Working yourself backwards

- As before. First solve the post disaster problem. Get the shadow prices

Post-disaster control (Copy of previous slide)

- There is no stochasticity. Simply solve the following problem:

$J(t,x|\tau = t)e^{rt} = \max_u \; e^{rt} \int_t^{\infty} U(u,y)e^{-rs}ds$ subject to $y(0) = x$ and $dy/dt=f(y,u)$.

- This looks like our old friend the embedded problem.
- Here t and x are arbitrary. The indicate the possible state(s) of the world after a disaster.
- The notation $J(t,x|\tau = t)$ indicates that J is evaluated conditional on τ happening at time t.

Pre Disaster Solution

- Define the risk Augmented Hamiltonian

$$H = f(x,u) + \mu g(x,u) + \lambda(x)(J(t, x + q(x) \mid \tau = t) - J(t,x))$$

- Then apply the maximum principle
- Note: λ is here the hazard rate and μ is the shadow price

Applying the Maximum Principle

- Find the control that maximizes the Hamiltonian and use d

$$u = \arg\max_{y}(H)$$

$$\dot{\mu} = r\mu - \frac{\partial H}{\partial x} = r\mu - f'_x(x,u) - \mu g'_x(x,u) + \lambda(x)\left(\frac{\partial}{\partial x} J(t,x) - \frac{\partial}{\partial x} J(t, x + q(x) \mid \tau)\right) + \lambda'(x)(J(t,x) - J(t, x + q(x) \mid \tau)).$$

- Compared to the case with exogenous risk we have an additional term. If risk is exogenous, we have that $\lambda'(x) = 0$

A closer look at dμ/dt

$$\dot{\mu} = r\mu - \frac{\partial H}{\partial x} = r\mu - f'_x(x,u) - \mu g'_x(x,u) + \lambda(x)\left(\frac{\partial}{\partial x} J(t,x) - \frac{\partial}{\partial x} J(t, x + q(x) \mid \tau)\right)$$
$$+ \lambda'(x)(J(t,x) - J(t, x + q(x) \mid \tau)).$$

- As in the exogenous risk case, we have that

$$\frac{\partial}{\partial x} J(t,x) = \mu(t) \quad \frac{\partial}{\partial x} J(t, x + q(t,x) \mid \tau) = \mu(t \mid t, x + q(x))(I^n + q'_x)$$

- We have J(x+g(x)|τ) from the post-disaster problem. But what about J(x)?

I turns out that:

- J(x) = z is given by:

$$\dot{z} = rz - f(x,u) + \lambda(x)(z - J(t, x + q(t,x)))$$

- So we can rewrite dμ/dt as:

$$\dot{\mu} = r\mu - \frac{\partial H}{\partial x} = r\mu - f'_x(x,u) - \mu g'_x(x,u)$$
$$- \lambda(x)(\mu(t \mid t, x + q(x))(I^n + q'_x) - \mu) - \lambda'(x)(J(t, x + q(x) \mid \tau = t) - z)$$

- Additional transversality condition z(T) = 0 (Why?)

A pedagogic problem

- After inserting for the optimal u, we have three differential equations in x, μ and z. Tough to do.
- In fact, I know of no application where one can find a closed form solution or even a reasonably manageable steady state.
- Numerical methods are all that is left. Oh, and general statements.

The Problem we are going to solve

$$\max E\left(\int_0^T \left(\alpha - \frac{1}{2}\left(u^0 - u\right)^2\right) e^{-rt} dt\right)$$

$$s.t.\ \dot{x} = u - \delta x,\ x(0) \text{ given}, \dot{\alpha} = 0, \alpha(0) = 0.$$

τ random variable with hazard rate $\mu^0 x(t)$.

$$\alpha(\tau^+) - \alpha(\tau^-) = -K,$$

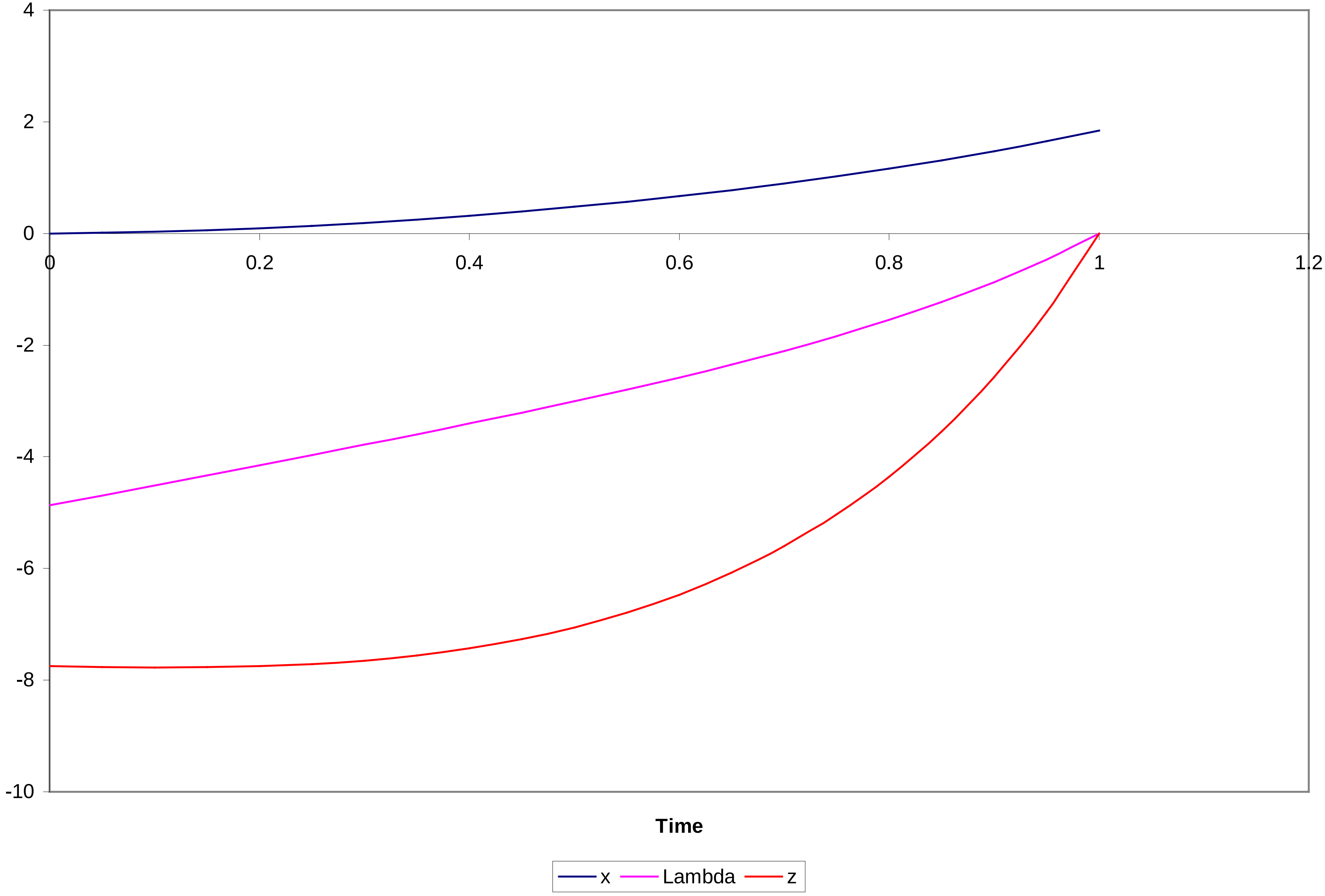
The post event solution

- Really simple. No more damage from *x* so post event shadow price is zero.
- This implies that $u = u0$ for all $t > \tau$.
- $J(x|t) = -K/r$

The pre-event problem

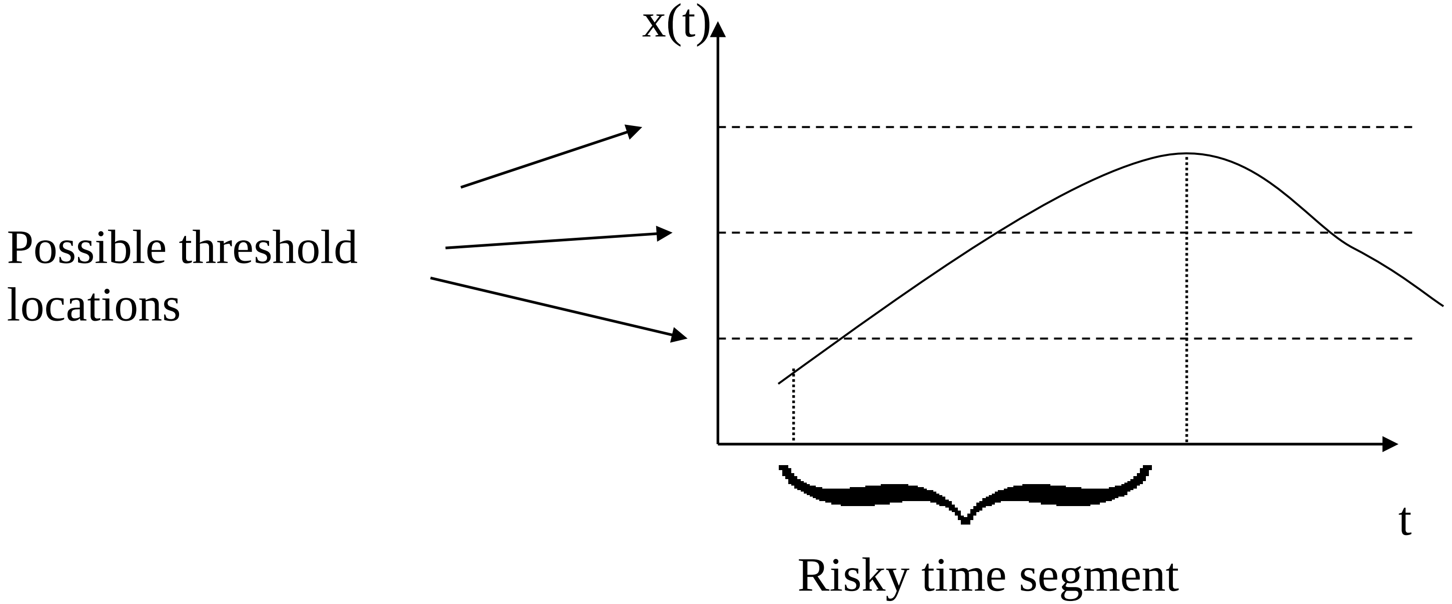
- Hamiltonian given by
- $H = \alpha - \frac{1}{2}(u^0 - u)^2 + \lambda(u - \delta x) + \mu \times x \times (-K/r - z)$
- Apply maximum principle to get:
- $u = \text{argmax } H = u^0 + \lambda$
- $d\lambda/dt = (r+\delta)\lambda + \mu x \lambda - \mu(-K/r - z)$
- $dz = rz + \frac{1}{2}(u^0 - u)^2 - \mu(-K/r - z)$
- Transversality conditions $\lambda(T) = z(T) = 0$

The results

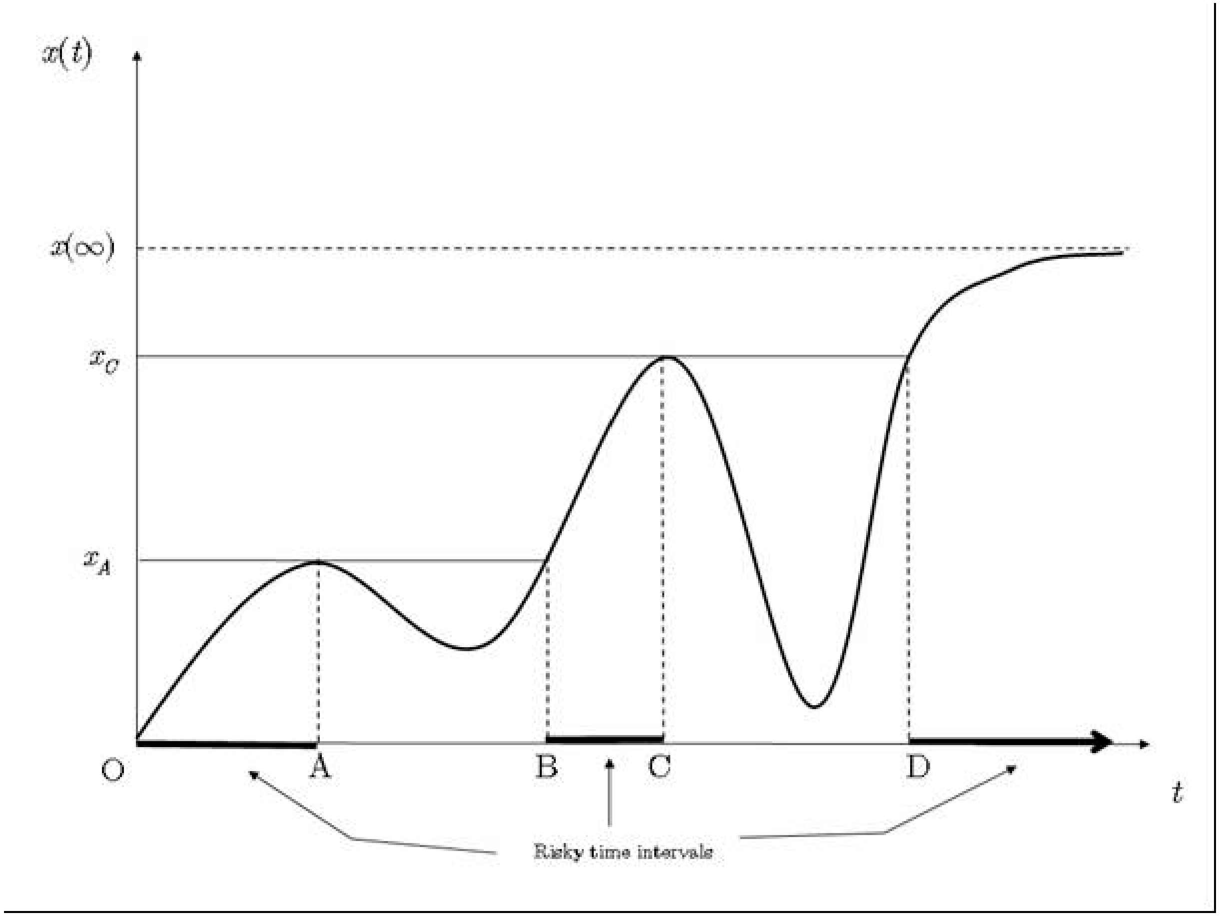


Endogenous, state-space distributed risk – AKA stochastic threshold

- We move a state variable through time. If we move in the "right" direction we may trigger a threshold effect.



Better illustration



Preliminary math.

- We all know the chain rule.

$$F(x)=U(g(x)) \rightarrow F'(x)=U'(g(x))g'(x)$$

- We should also know that

$$\int f(x(t))x'(t)dt = \int f(x)dx$$

$$\int_0^t f(x(t))x'(t)dt = \int_{x(0)}^{x(t)} f(x)dx$$

Let f(x) be a distribution

- If the event x = x' is distributed f(x) over [a, b] and…
- x(t) is such that x'(t) ≥ 0 for all t and Range[x(t)] = [a, b]
- Then the cdf for the event x = x' and the event x(τ) = x are interchangeable. The pdf for the event is distributed over time is f(x(t))x'(t).
- Regardless of whether Range[x(t)] = [a, b], the hazard rate is

$$\mu(x(t)) = \frac{f(x(t))\dot{x}(t)}{1 - F(x(t))}$$

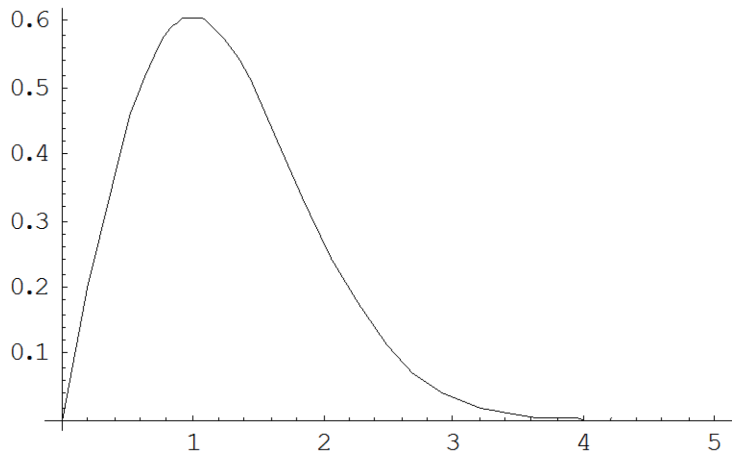
The hazard rate continued

- Note how the hazard rate depends on the rate of increase in x. dx/dt = 0 implies that the hazard rate is zero.

$$\mu(x(t)) = \frac{f(x(t))\dot{x}(t)}{1 - F(x(t))}$$

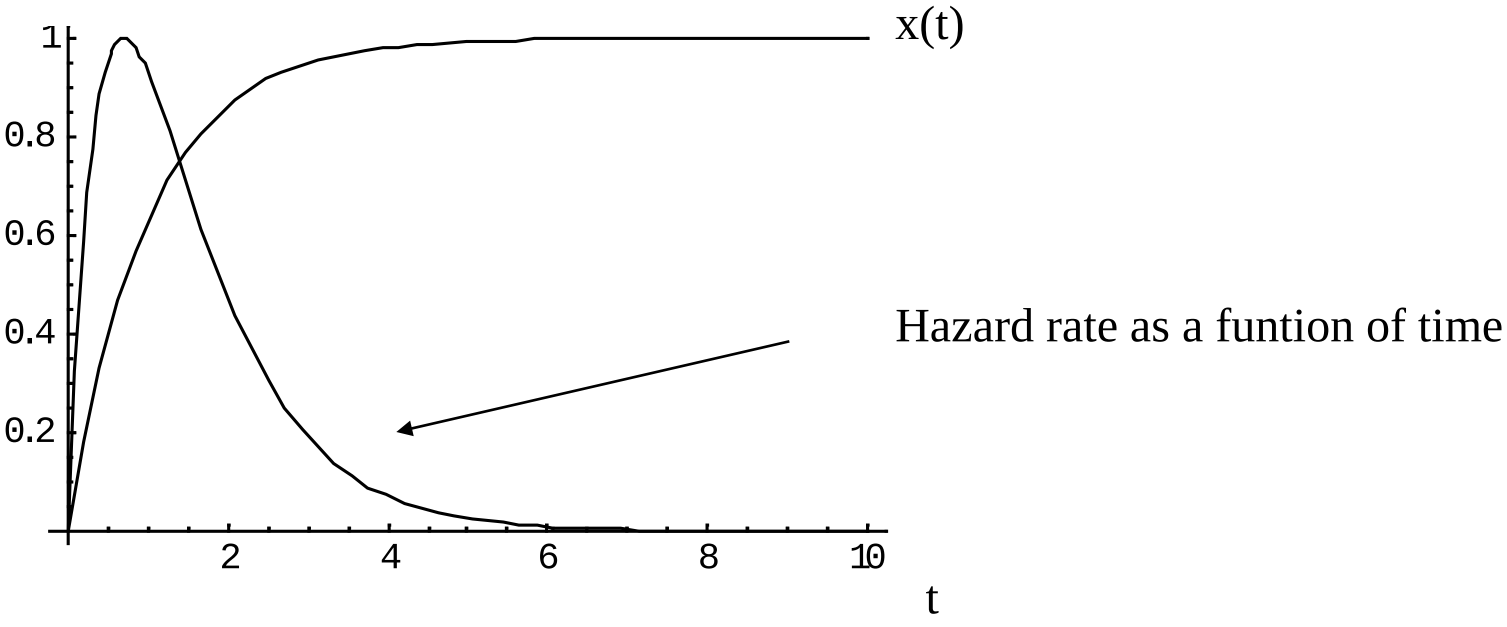
Example. The hazard rate in state-space is linear

- The hazard rate is linear in state space. This is the pdf:



- Assume that $x'(t) = -ax(t) + b$

However, the hazard rate in time behaves differently



Properties of the hazard rate

- If we move away from the threshold, the hazard rate is zero

$$\mu\left(x(t)\right)=\frac{f\left(x(t)\right)}{1-F\left(x(t)\right)}\max\left(\dot{x},0\right)$$

- If we let x take values that have happened before, then the hazard rate is also zero.

Optimal control

- Use exactly the same conditions as with endogenous time distributed problems, but with modified hazard rate.
- Technical note: Previously the hazard rate depended only on state variables. Now it will in general also depend on control variables. (Why?) This does not matter as long as we can restrict ourselves to look at continuous controls,

Example

- Same as before. However, this time we will consider a threshold problem.

The Problem we are going to solve

$$\max E\left(\int_0 \left(a - ax - \frac{1}{2}\left(u^0 - u\right)^2\right) e^{-rt} dt\right)$$

$s.t.\ \dot{x} = u - \delta x, x(0)$ given, $\dot{a} = 0, a(0) = 0.$

$\bar{x}$ random variable with hazard rate μ^0. Implies $\bar{x}$

is exponentially distributed over $\left[x(0), \infty\right)$

Event triggered by $x(t) = x$

$a(t^+) - a(t^-) = -K,$

Solution strategy

- Solve recursively
- First compute optimal solution after crossing the threshold
- Use this solution to compute optimal paths prior to crossing the threshold

The post event solution

$$u\left(s \mid t, \bar{x}\right) = u^0 - \frac{a}{c\left(r + d\right)}$$

$$m\left(s \mid t, \bar{x}\right) = - \frac{a}{r + d}$$

$$x\left(s \mid t, \bar{x}\right) = \frac{u^0 c(r + d) - a}{cd(r + d)} \times \left[1 - e^{-d(s-t)}\right] + \bar{x} e^{-d(s-t)}$$

$$J\left(t, \bar{x}\right) = - \frac{a}{r + d}\bar{x} + \frac{a^2 - 2acu^0(r + d)}{2cr(r + d)^2} - \frac{G}{r}$$

The pre-event problem

- Hamiltonian given by
- $H = \alpha - \frac{1}{2}(u^0 - u)^2 + \lambda(u - \delta x) +$

  $\mu \times (u - \delta x)(-K/r - z)$
- Apply maximum principle to get:
- $u = \text{argmax } H = u^0 + \lambda - \mu(K/r + z)$
- $d\lambda/dt = ax + (r+\delta)\lambda + \mu(u - \delta x)\lambda + \mu\delta(J(x|\tau) - z)$
- $dz = rz + \frac{1}{2}(u^0 - u)^2 - \mu(u - \delta x)(J(x|\tau) - z)$

Note the difference

Hazard rate now directly affects control!

Derivative of hazard rate

Optimal u from maximization

Analytics

- One can actually compute steady states here. In steady state x'(t) is zero so lots of stuff disappears. To whit:
- $u = u^0 + \lambda - \mu(K/r + z)$
- $d\lambda/dt = - a + (r+\delta)\lambda - \mu\delta(K/r + z)=0$
- $dz/dt = rz + ax + ½(u^0-u)^2 = 0$
- $dx/dt = u – \delta x = 0$

Steady state solution

$$x^{ss} = \lim_{t \to \infty} x(t) = \frac{u^0}{d} - \frac{a}{cd(r+d)} + \frac{1}{dl}\left((r+d) - \sqrt{(r+d)^2 + 2G\frac{l^2}{c}} \right)$$

$$u^{ss} = \lim_{t \to \infty} u(t) = u^0 - \frac{a}{c(r+d)} + \frac{1}{l}\left((r+d) - \sqrt{(r+d)^2 + 2G\frac{l^2}{c}} \right)$$

Note the following: K = 0 implies x and u equal post-disaster steady state
Otherwise K > 0 implies small x and u. K large enough implies x and u less than zero in steady state.

Some economic implications

- We all know that the discount rate is super important for optimal climate policy
- Or is it?
- Think of a deterministic threshold problem. Does the decision of where to stop depend on the discount rate?

Examining the solution

$$x^{ss} = \lim_{t \to \infty} x(t) = \frac{u^0}{\delta} + \Delta x_a^{ss} + \Delta x_G^{ss} \text{ where}$$

$$\Delta x_a^{ss} = -\frac{a}{c\delta(r+\delta)}, \quad \Delta x_G^{ss} = \frac{1}{\delta\lambda}\left((r+\delta) - \sqrt{(r+\delta)^2 + 2\Gamma \frac{\lambda^2}{c}} \right)$$

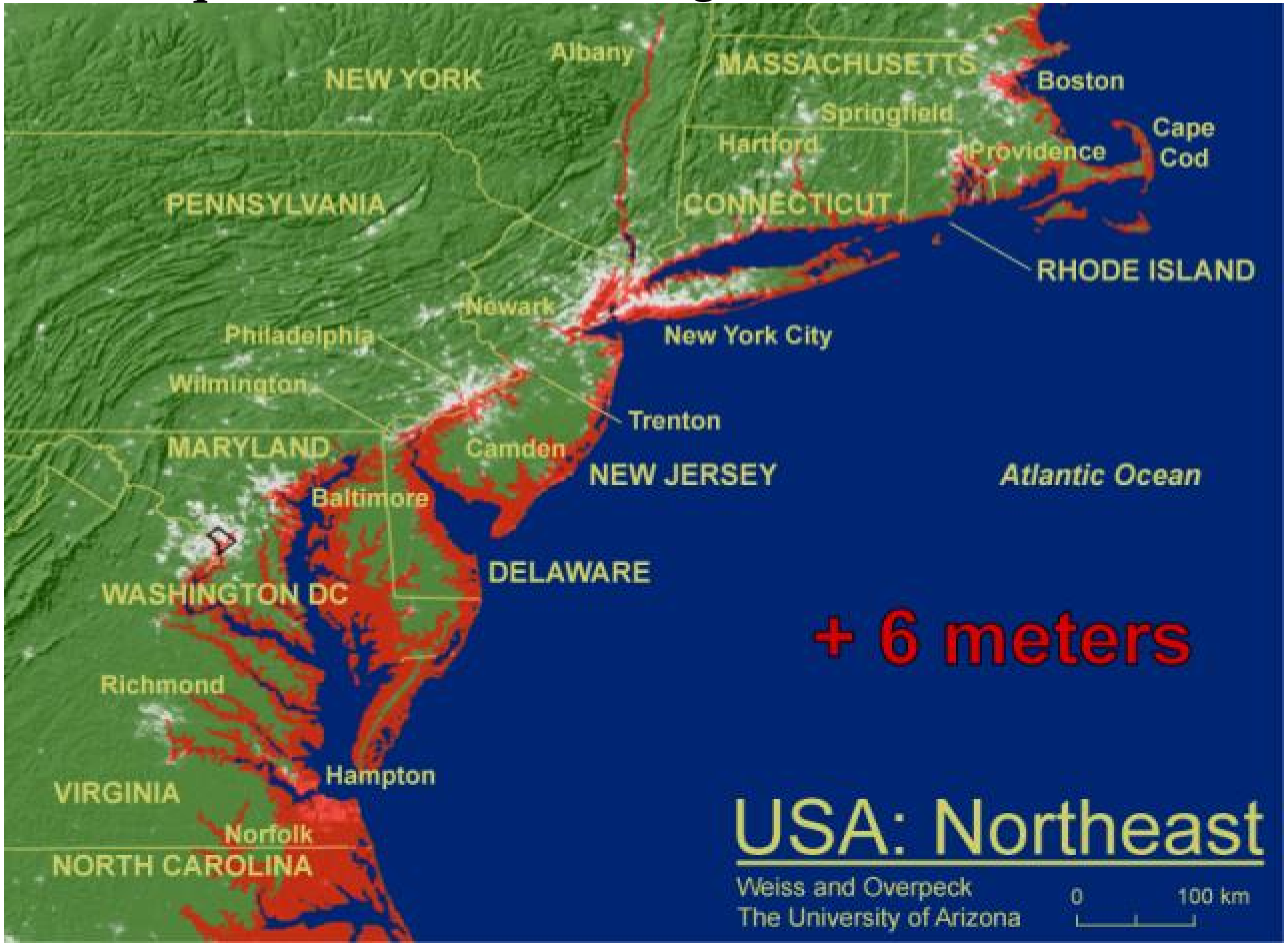
A caveat

- This solution assumed that $x(0) \leq x_{ss}$. Otherwise we get negative hazard rate. What happens if we have $x(0) > x_{ss}$
- Answer: Freeze x at x(0) indefinitely.

Application – Disintegration of the Western Antarctic Ice Sheet

- Oppenheimer (1998) estimates that a WAIS disintegration could increase sea levels by as much as 4-6 meters.
- Oppenheimer evaluates that there is a threshold temperature increase above pre-industrial levels where this event could occur. This threshold lies in the range 2.5° to 8° degrees Celsius.
- What does this imply for human welfare?

Consequences of WAIS disintegration



Considerable impact on real state markets.

For some areas such as Bangladesh, the picture is even grimmer

Some relevant geoscience

There are several GHGs that contribute to global warming. Methane (1/5) and Carbon (3/5) being the most important. Because of different retention periods in the atmosphere they must be treated separately.
Let *c* be carbon and *m* be methane. Then there is a threshold defined by:
$ac(t) + bm(t) = A + \varepsilon$.

The functions *c* and *m* are determined by

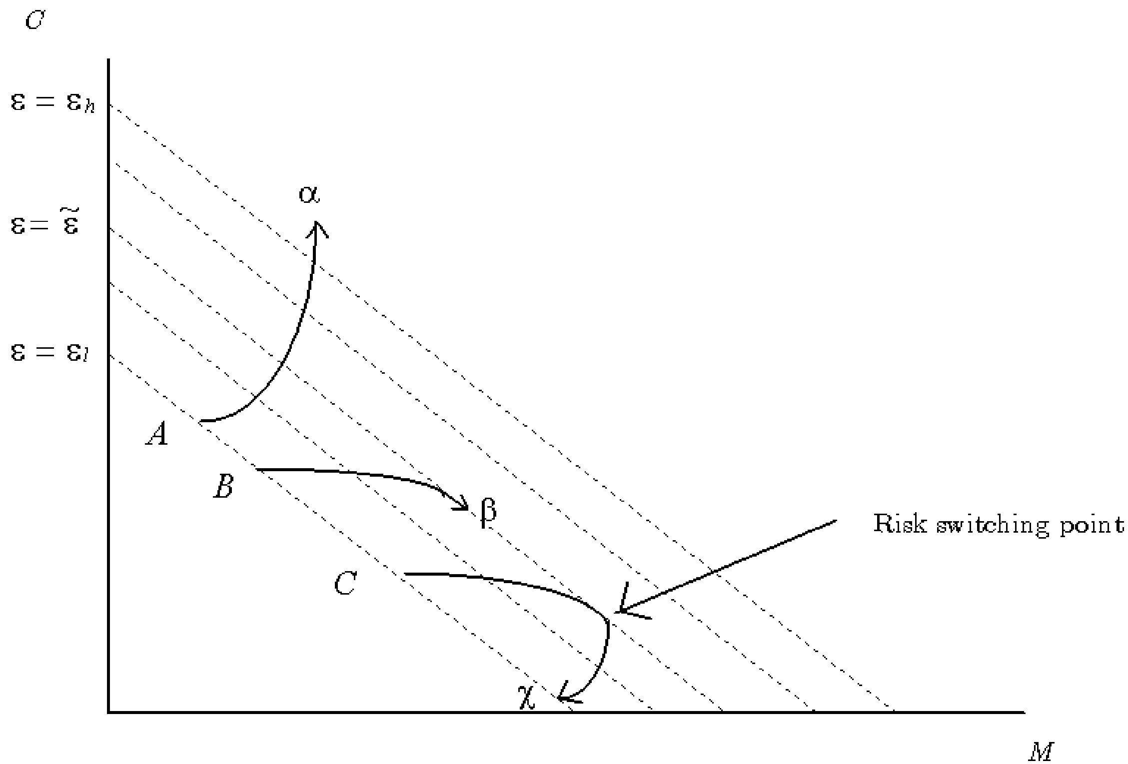
$$\dot{c} = u_c - \delta_c c$$

and

$$\dot{m} = u_m - \delta_m m$$

$A + \varepsilon$ is the temperature threshold, ε is a random variable. Emissions are given by *ui*

Possible paths of the system



The Main Technical Problem.

How to translate the event that the threshold is crossed from a probability distribution in State-space to a probability distribution over time?

Answer: Change of variable techniques from integration theory.

The Economy

The instantanous cost of emission reduction

$$k_i(u_i) = \frac{K_i}{2}\left(u_i^0 - u_i\right)^2, \quad i = i, m$$

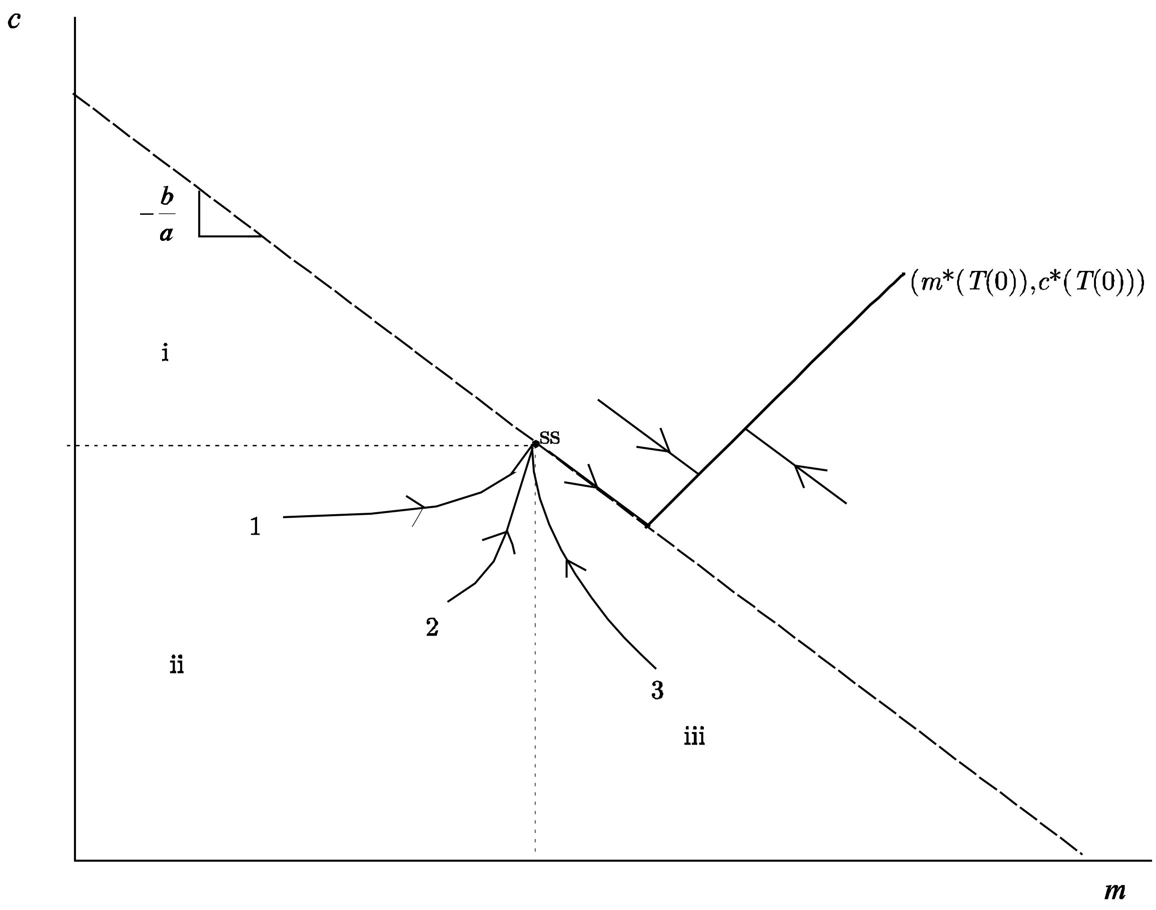
The cost of crossing Wais disintegration: $\gamma = G$ if threshold is crossed, $\gamma = 0$ otherwise

We can now state society's optimization problem as follows:

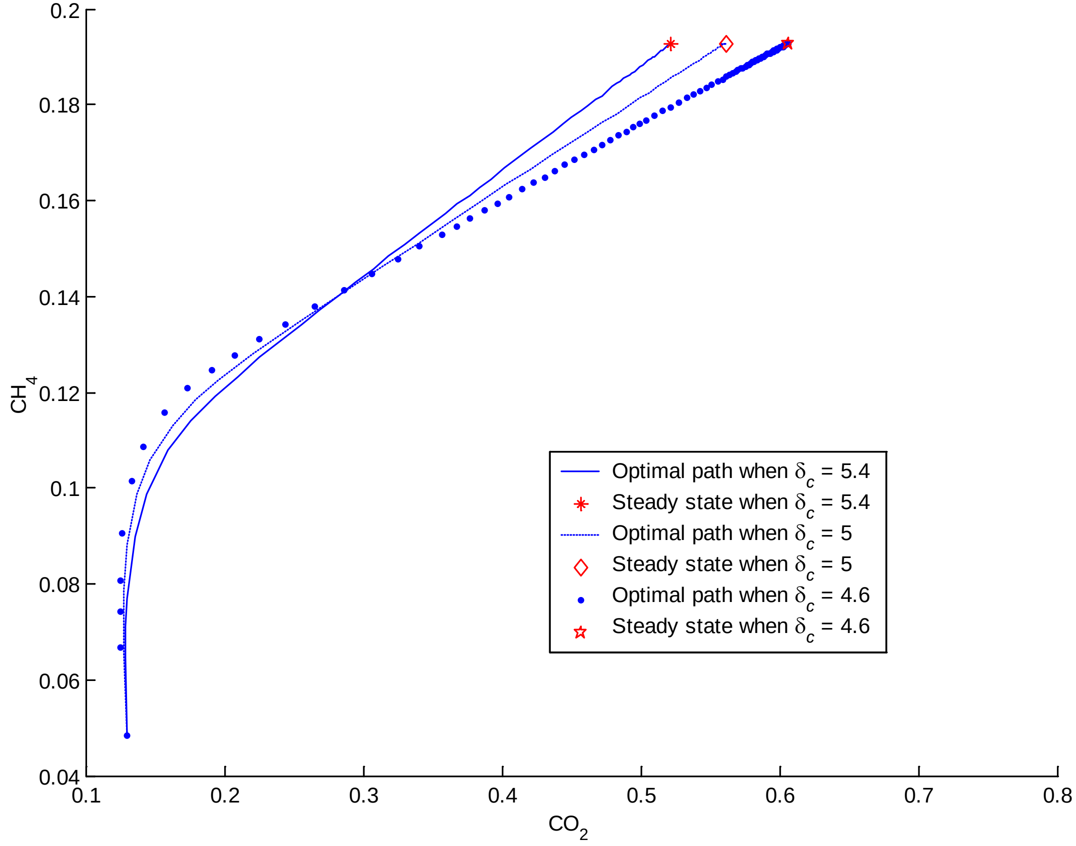
$$\max_{u_c, u_m} E\left(\int_0^\infty \left(-\gamma - \frac{K_c}{2}\left(u_c^0 - u_c\right)^2 - \frac{K_m}{2}\left(u_m^0 - u_m\right)^2\right)dt\right)$$

Steady states are calculated in the paper

Optimal Paths



Some numerical comparative dynamics



The Economics of the Thermohaline Circulation – A Problem with two Thresholds

Some Unpleasant Facts and Possibilities

- Historical Geophysical data suggests that the Thermohaline Circulation has been disrupted in the past.
- Scientific results suggests that this may be triggered by Global Warming.
- The consequences, although uncertain, will be a bugger. They include but are not limited to:
  - Regional disruptions in weather patterns
  - Permanent regional climate change
  - May trigger other global catastrophic events.

Some Science

- Global Warming will affect ocean temperatures and salinity which are the main drivers of the thermohaline circulation.
- THC disruption depends on temperature levels and the rate of change in temperature.
- If temperature levels or rates of temperature change exceed certain thresholds a shutdown may occur. The location of these thresholds are unknown.

A Simple Model of Global Warming

c – Atmospheric carbon concentration

F – Increase in global temperature average

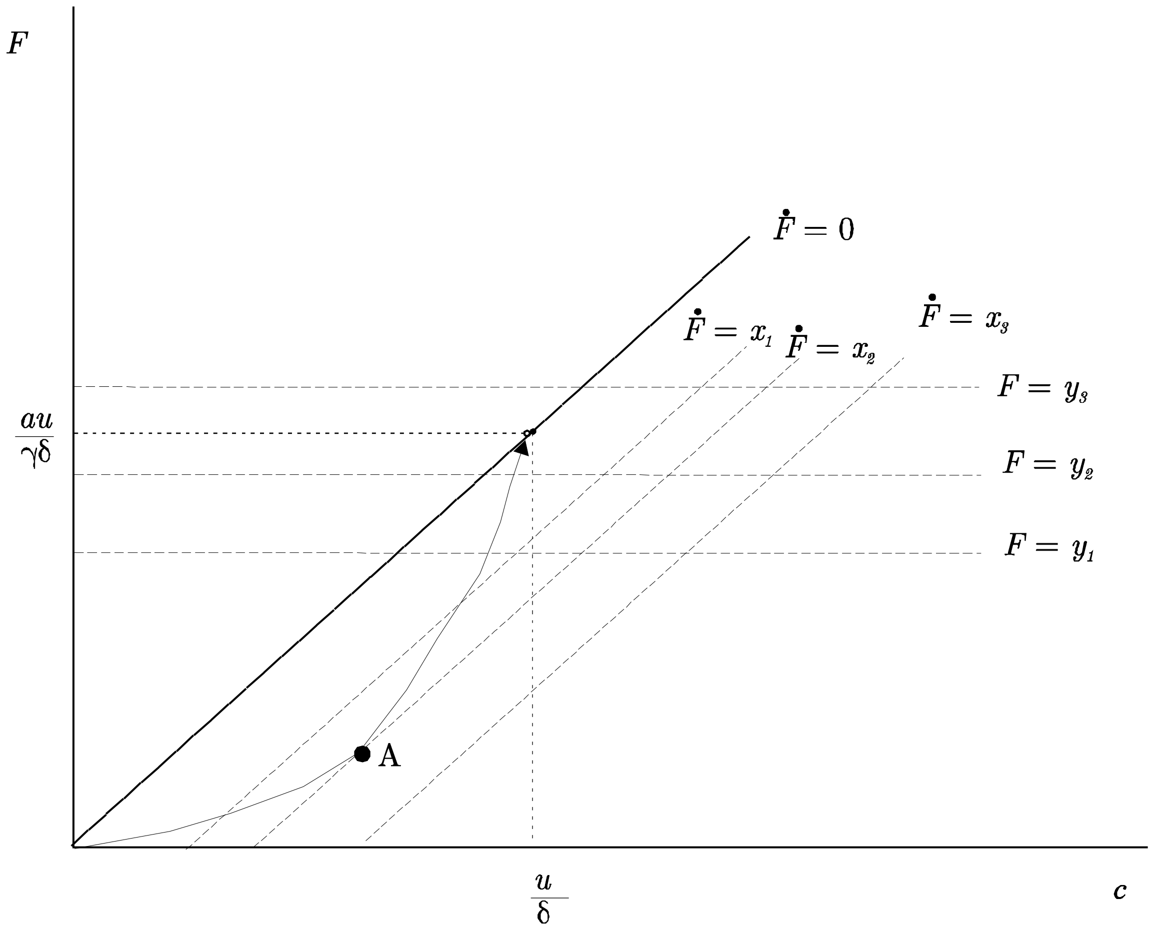
u – CO_2 *emissions*

$$\dot{c} = u - \delta c$$

$$\dot{F} = \alpha c - \gamma F$$

If F or $\dot{F}$ increase above certain thresholds, the THC collapses.

Illustration of the Risk Structure



The Economic Problem(s)

- What is the optimal CO_2 emission pattern in the presence of this risk?
- How to optimize such a system? Piecewise Deterministic Optimal Control!
- Some technical issues must be resolved before optimization can be done.
  - The events distributed in state space must be transformed so that they are distributed over time.
  - A distribution for the minimum of these events must be derived.

The Optimization Problem

$$\max_u E\left(\int_0^\infty \left(-\gamma - \frac{K}{2}\left(u^0 - u\right)^2\right)dt\right)$$

Climate cost. 0 if THC is OK, *G* if THC collapses

Cost of reducing emissions below unregulated emission level u^0.

The Optimization Problem contnd.

Subject to the differential equations and two concurrent stochastic processes.

As long as both F *and d*F/d*t are increasing the hazard rate is the sum of two hazard rates. When only* F *is increasing, only this hazard rate is active*

The Hamiltonian for the present problem

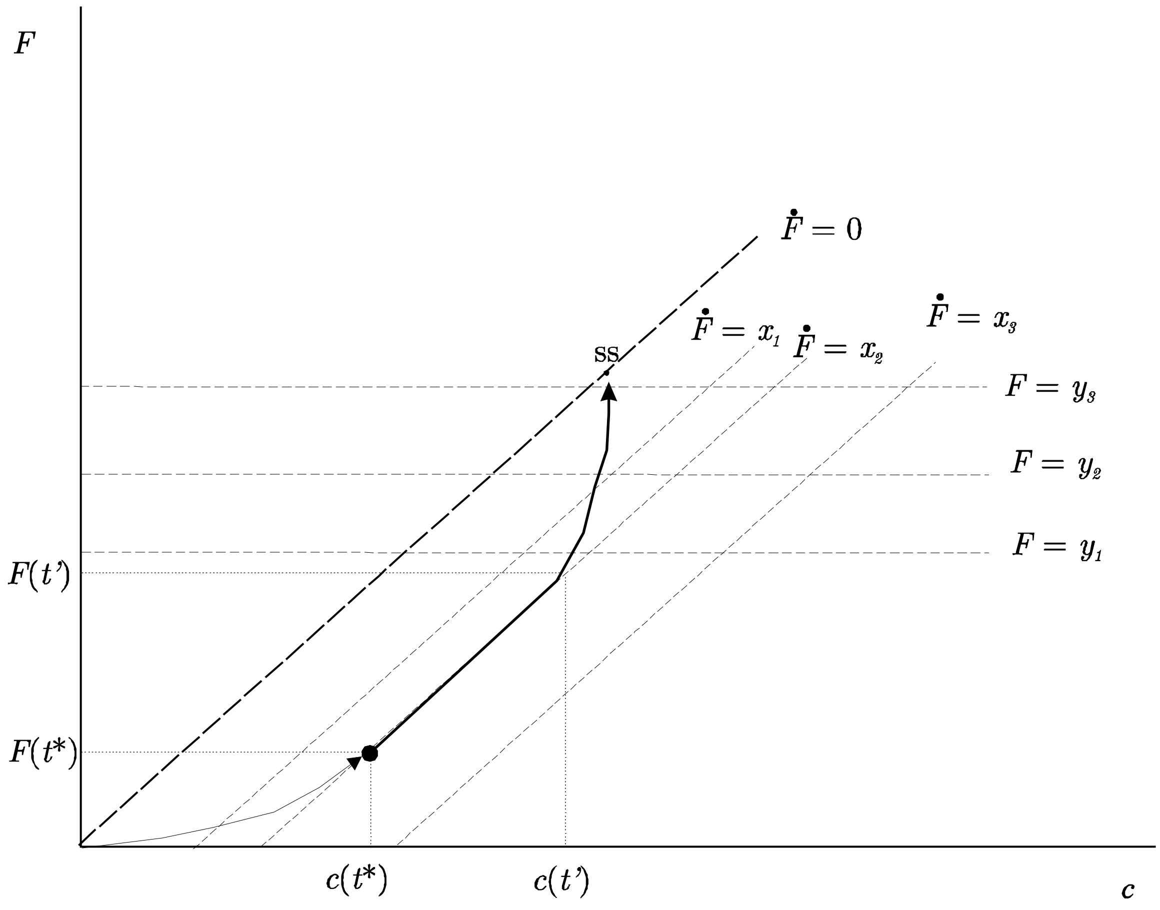
First terms are the standard Hamiltonian

$$H = -\gamma - \frac{K}{2}\left(u^0 - u\right)^2 + \chi\left(u_c - \delta_c c\right) + \mu\left(u_m - \delta_m m\right)$$

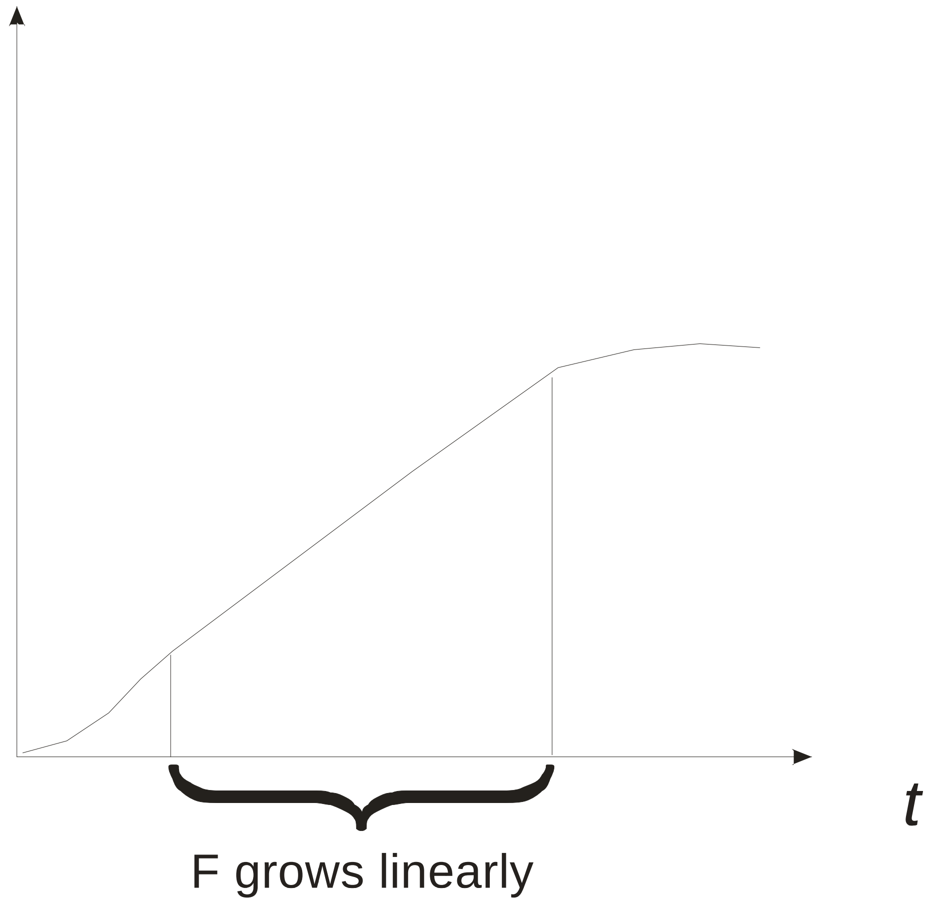
$$+ \lambda(m,c)\left(J\left(c,t \mid \tau\right) - z\right)$$

This term is the expected loss/gain from disaster in the interval (*t*, *t* + *dt*)

The Optimal Path



Optimal Time path for Temperature



Optimal stopping

- We have a risky process and we wonder when to stop.
- In deterministic problems, we use H = 0. Here too.
- In Finite time:

Max $E\{\int^T (U(x,u)e^{-rt})dt + h(x(\tau^-))e^{-r\tau} + G(x(T))e^{-rT}$

subject to $dx/dt=f(x,u)$ for all $t \neq \tau$

$x(\tau^+)-x(\tau^-) = g(x(\tau-))$

τ distributed $\mu e^{-\mu t}$ over $[0, \infty)$

A quick word about the value function

- If we have a scrap value, then we have that $J(T,x(T))$ = Scrapvalue.
- Intuitively obvious, but needs to be pointed out

Optimal Stopping (Eksogenous Risk)

- Apply the maximum principle to the Hamiltonian:

$H = U(u,x) + \lambda f(u,x) + \mu(t)(h(x)e^{-rt} + J(x+g(x)|\tau) - J(x))$

- u =argmax H
- $d\lambda/dt = r\lambda - \partial H/\partial x \quad \lambda(T) = \partial G(x(T))e^{-rT}/\partial x$
- $dx/dt=f(x,u)$

Optimality Conditions

- The Stopping Condition

$$U(u,x) + \lambda f(u,x) + \partial\, G(x(\tau^{-}))e^{-r\tau}/\partial t$$

$$-\ \mu(t)(h(x)e^{-rt} + J(x+g(x)|\tau) - G(x(T))e^{-rT}) = 0$$

Optimal harvesting of illegal drugs

- A drug producer grows marihuana. At optimally chosen time T he sells the stuff at a price q, unless…
  - He gets raided by the cops, in which case he receives a punishment –c per weight unit of weed
  - The local college boys finds the farm and nicks a fraction 1 – b of the drugs. The farmer then sells the remaining.

Mathematical formulation

- Equations of motion

$$\dot{x}_1 = at^{a-1} - x_2 at^{a-1}$$
$$\dot{x}_2 = 0$$
$$x_1(0) = x_2(0) = 0$$

- Maximization problem
- max $E((\lambda_1(-cx_1e^{-r\tau})+\lambda_2 bqx_1e^{-r\tau})/\lambda) + qx_1(T)e^{-rT}$

The Jump

- x_2 is a just a way of saying that if τ occurs, the dope stops growing

$$x_1(\tau_1^+) - x_1(\tau_1^-) = -x_1(\tau_1^-)$$
$$x_2(\tau_1^+) - x_2(\tau_1^-) = -x_2(\tau_1^-) + 1$$

The Maximum Principle

$$\dot{x}_1(s) = as^{a-1}$$
$$\dot{p}_1(s) = (\lambda_1 + \lambda_2)p_1(s) + \lambda_1 ce^{-rs} - \lambda_2 qbe^{-rs}$$

- We only need $p(T) = qe^{-rT}$.

The Stopping condition

H $= qe^{-rT}aT^{a-1} - rqT^{a}e^{-rT} + (\lambda_1 + \lambda_2)(-qT^{a}e^{-rT}) + (\lambda_1 bq - \lambda_2 c)T^{a}e^{-rT} = 0$

Which can be solved and yield:

$$T = \frac{aq}{qr + (\lambda_1 + \lambda_2)q + (\lambda_1 c - \lambda_2 bq)} < \frac{a}{r}$$