THE LAW OF DIMINISHING RETURNS

and other variations on the

WEIERSTRASS THEOREM

Rolf Färe

WEIERSTRASS THEOREM

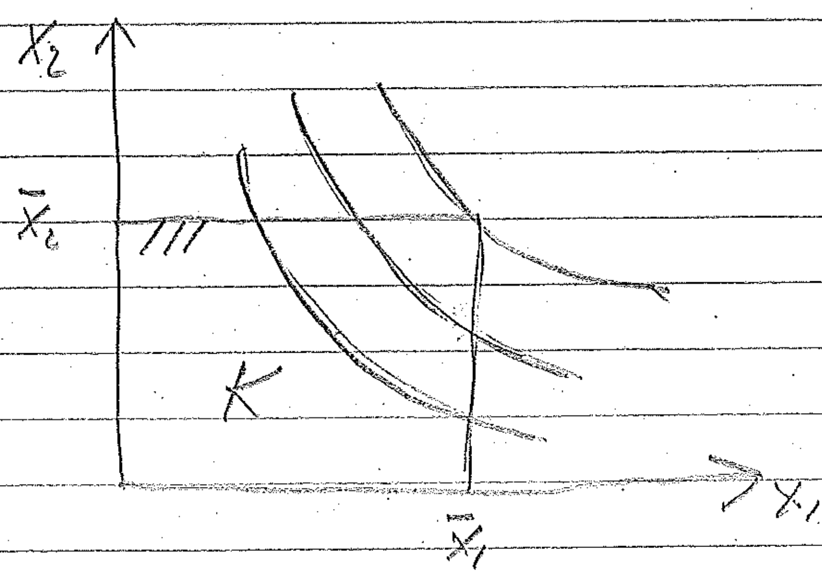
$F: K \subseteq \mathbb{R}^2_+ \to \mathbb{R}_+$

F continuous, K compact

then

$$\sup_{x_1, x_2 \in K} F(x_1, x_2) < +\infty$$

(actually sup is a max)



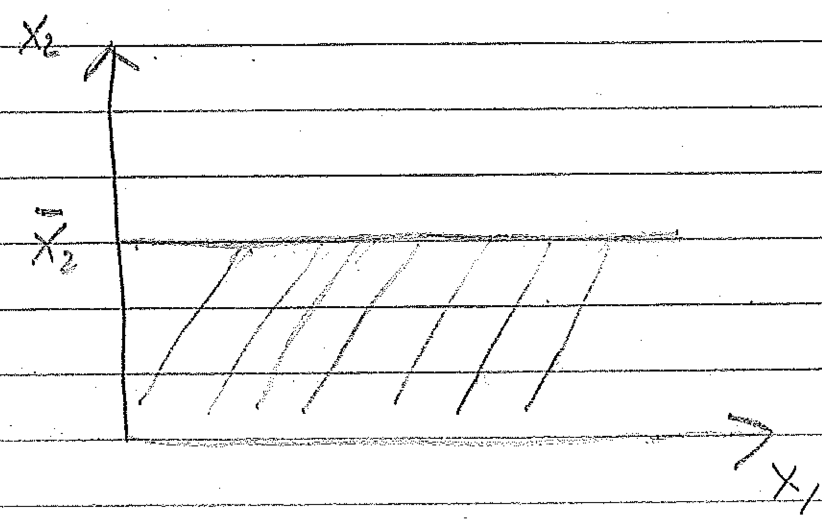
THE LAW OF DIMINISHING RETURNS

Anne Robert Jaques Turgo (1844)

Shephard (1970, p. 42) "... if the law did not hold the output of every piece of land could be unbounded..."

$$\sup_{\substack{x_1 \geq 0 \\ x_2 \leq \bar{x}_2}} F(x_1, x_2) < +\infty$$

(x_2 is Limitational)



Domain not bounded

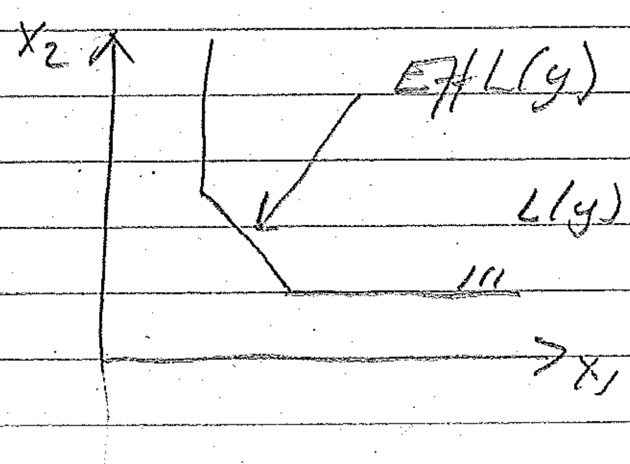
Examples

i) $$\sup_{\substack{x_1 \geqq 0 \\ x_2 \leqq \bar{x}_2}} x_1^{\alpha} \cdot x_2^{1-\alpha} = +\infty$$

ii) $$\sup_{\substack{x_1 \geqq 0 \\ x_2 \leqq \bar{x}_2}} \min\{x_1, x_2\} \left(= \bar{x}_2\right) < +\infty$$

$$L(y) = \{x : F(x) \geqq y\}$$

$$\text{Eff } L(y) = \{x \in L(y) : x' \leqq x \Rightarrow x' \notin L(y)\}$$



Eff L(y) bounded ; Activity Analysis

Explaining the Law

X_2 is essential if $F(x_1, 0) = 0$

Thm: x_2 is limitational $\Longleftrightarrow$ x_2 is essential

i) bounded eff subsets

ii) continuity

iii) constant returns to scale

Färe and Primont (2002) Shephard (1970)

($\Rightarrow$) assume x_2 is not essential

then there is a $(x_1, 0)$ such that

$F(x_1, 0) > 0$, by CRS

$F(\lambda x_1, \lambda 0) = \lambda F(x_1, 0) > 0$, $\lambda > 0$

thus $\lim_{\lambda \to \infty} F(\lambda x_1, 0) = +\infty$

Thus

X_2 is not limitational

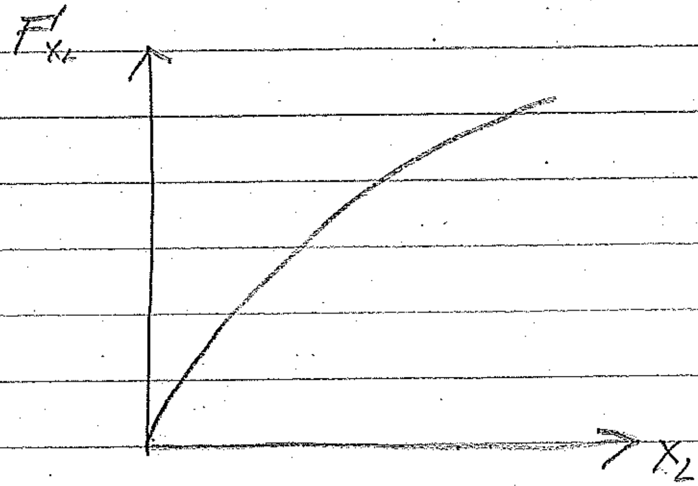
i.e., X_2 is essential.

Marginal Productivity

$$\frac{\partial F(X_1, X_2)}{\partial X_2} = F'_{X_2}$$

$$X_2 \text{ essential} \Longrightarrow \lim_{X_2 \to +\infty} F'_{X_2} \to 0$$

A. Irmen and A. Maußner SIFO 5126 (2014)



LAW OF THE MINIMUM

" Justus von Liebig's Law of the Minimum states that yield is proportional to the amount of the most limiting nutrient, whichever nutrient it may be."

∴ Which nutrient is the most "limitational" ?

Pear Trees example

Wang, Färe, Seavert J. Amer Soc
Hort Sci (2006)

Outputs : various pear size/quality

Inputs : trunk area, micro nutrients

Activity Analysis

max py

$$\text{s.t. } y \in \underbrace{(\quad)\quad(\quad)}_{\text{Activity Analysis model}}$$

Eff $L(y)$ bounded

By allowing essential inputs to be free we computed which input had the strongest binding effect

$$\sup_{\substack{x_1 \geq 0 \\ x_2 \leq \bar{x}_2}} F(x_1, x_2) \in \mathbb{R}_+$$

$$\sup_{\substack{x_2 \geq 0 \\ x_1 \leq \bar{x}_2}} F(x_1, x_2) \in \mathbb{R}_+$$

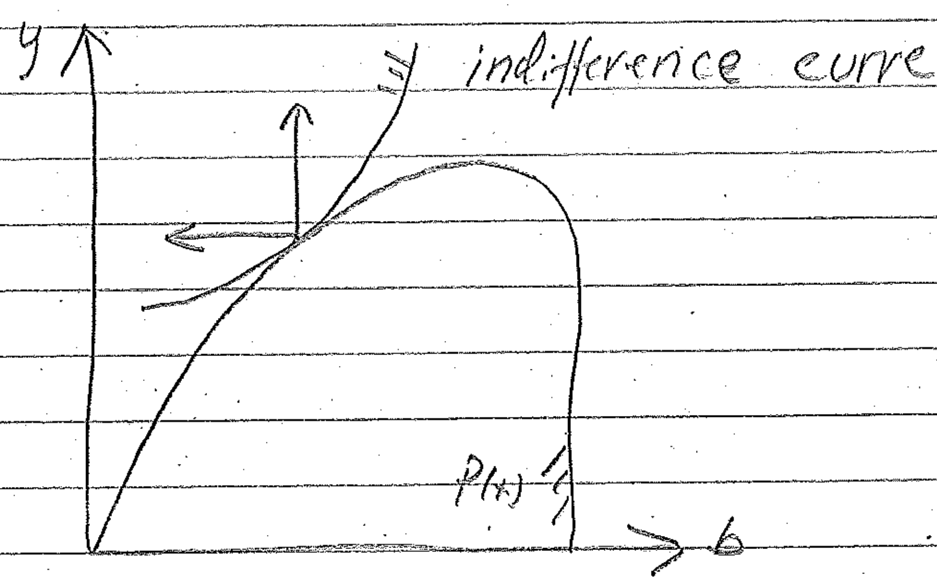
REGULATION AND UNINTENDED CONSEQUENCES

$y \geqq 0$ desirable output

$b \geqq 0$ undesirable or bad output

The Output Set

$$P(x) = \{(y,b) : x \text{ can produce } (y,b)\}$$

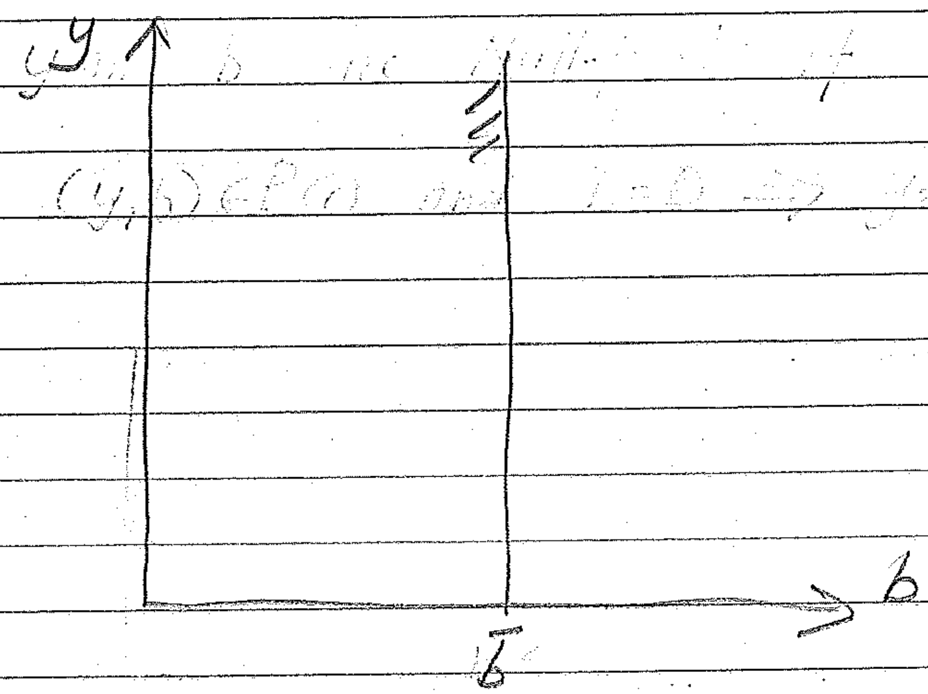


Question: If b is regulated say $b \leq \bar{b}$, does this restrict good output?

Notions:

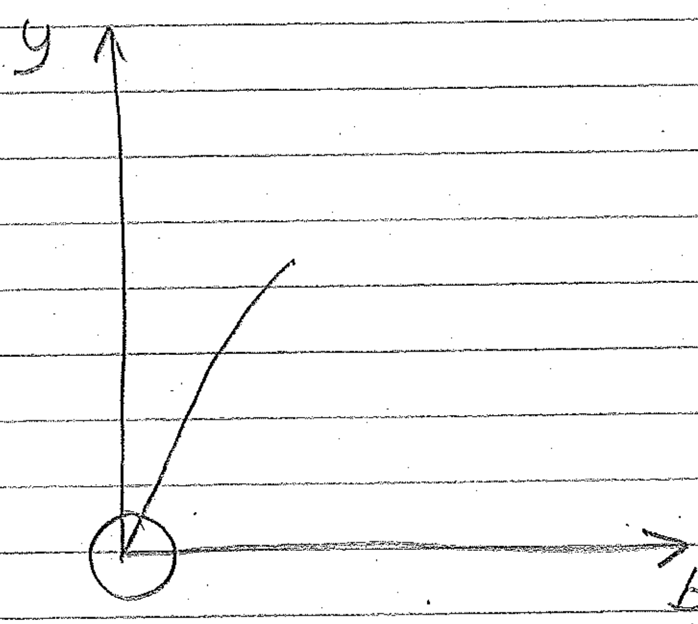
b is Output Limitational if for $\bar{b} \geq 0$,

$$\sup_{\substack{x \in \mathbb{R}^N \\ b \leq \bar{b}}} \{ y : (y, b) \in P(x) \} < +\infty$$



y and b are null joint if

$(y,b) \in P(x)$ and $b=0 \Rightarrow y=0$



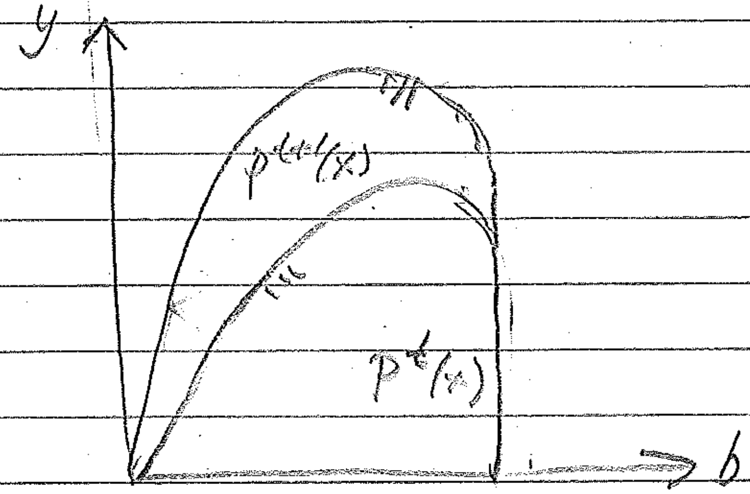
Thm: b is output limitational $\Leftrightarrow$

(y,b) are null-joint

i) CRS $P(\lambda x) = \lambda P(x)$, $\lambda > 0$

Offset the unintended consequences

by technical change



WHICH BAD IS WORST

Activity Analysis $\{(x^k, y^k, b^k)\ k=1\ldots K \text{ obs}\}$

$$\max\ y$$

$$\text{s.t} \quad \sum_{k=1}^{K} z_k y_k \geq y$$

$$\sum_{k=1}^{K} z_k b_{k1} = b_1 \ (\text{free})$$

$$\sum_{k=1}^{K} z_k b_{kj} = b_{k'j} \quad j=2\ldots J$$

$$\sum_{k=1}^{K} z_k x_{kn} \leq x_n,\ n=1\ldots N \ (\text{free})$$

$$z_k \geq 0,\ k=1\ldots K$$

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