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Abstract

Discrete-time stochastic models of groundwater management have been extensively used for understanding a variety of issues in groundwater management for agriculture. Most models used suffer from two drawbacks: relatively simplistic treatment of extraction cost (remarked in many papers in the literature) and lack of important structural results (monotonicity of extraction in stock, concavity of the value function etc), even in simple models. Lack of structural properties impede both practical policy simulation (due to the lack of robustness) and clear understanding of the resulting models and underlying economics.

This paper provides a unifying framework for discrete-time stochastic groundwater models in two directions; first, the usual cost function is extended to encompass cases where marginal cost of pumping depends on the stock and second, the analysis here dispenses with assumptions regarding concavity of the objective function and compactness of the state space, using instead lattice-theoretic methods. With these modifications, a comprehensive investigation of which structural properties can be proved in each of the resulting cases is carried out. It is shown that, for some of the richer models, *more* structural properties may be proved than for the simpler model used in the literature.

In this set-up, convergence of the resource stock under the optimal policy typically follows from monotonicity of extraction in stock. This paper introduces to the resource and agricultural economics literature, an

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important method of proving convergence to a stationary distribution which does not require monotonicity. This method is of interest in a variety of renewable resource settings.

Keywords: Stationary distributions, Dynamic programming, Groundwater
JEL codes: Q25, C61, C62

1 Introduction

There is a substantial economic literature on groundwater management in a single-cell aquifer by a single user beginning with Burt (1964), described in Koundouri (2004). In this paper, we consider the problem of managing groundwater under random recharge in a single cell aquifer. There are two main modeling paradigms used in the literature: the first based on optimal control of the continuous-time model, e.g., Brown and Deacon (1972), Gisser and Sanchez (1980), Tsur and Graham-Tomasi (1991), Hellegers et al. (2001), Tsur and Zemel (2004), Zeitouni (2004), Roseta-Palma and Xepapadeas (2004), Rubio and Casino (2001) and the second based on dynamic programming formulation of the discrete-time model, e.g., Burt (1964, 1966, 1967, 1970), Provencher and Burt (1993) and Knapp and Olson (1995, 1996).

Most continuous time models, due to the specific assumptions made, yield explicit solutions. For instance, in the linear case, there are deterministic (Aggarwal and Narayan (2004)) or stochastic (Zeitouni (2004)) "bang-bang" solutions while in the linear-quadratic case (Roseta-Palma and Xepapadeas (2004)), there are analytical solutions possible. There have been very few continuous time models wherein structural properties have been derived from relatively few assumptions on the primitives. While several discrete-time dynamic programming formulations have been proposed in the literature, few structural properties for this problem have been proven. In fact, it is only Knapp and Olson (1995) who provide a first proof of the relatively straightforward property that extraction is increasing in current period stock.

Further, almost all of these formulations, both discrete- and continuous-time, make strong assumptions regarding the cost of extraction. In particular, two assumptions are made, usually implicitly: (a)marginal cost of pumping is independent of the quantity pumped (Chakravorty and Umetsu (2003) explicitly state this assumption) and (b) cost of pumping depends only on the beginning of period stock. Both of these are assumptions with little empirical support, even for (commonly assumed) unconfined aquifers. Yet, even with these assumptions, most structural properties of interest (see below) have not been proved.

Formally, we consider here the canonical discrete-time stochastic version of the model[1]. The main objective

[1]That this setting, for a wide variety of renewable resource problems, is similar to the one-sector stochastic growth model has been documented extensively (beginning at least with Mendelssohn and Sobel (1980)). Many resource problems (such as those on resource extinction, see Olson and Roy (2000), Mitra and Roy (2006) and references therein) are dealt with in the setting of the stochastic growth model. However, given that the model can be solved independently in a dynamic programming framework, we do not emphasize such a linkage here.

of this paper is to provide a unified treatment of the structural properties of the dynamic programming problem of groundwater extraction. The unification is both in terms of methods of proof used (relying on lattice-theoretic methods) as well as in extending the analysis to encompass a variety of extraction cost functions. The analysis here is in the spirit of Mendelssohn and Sobel (1980), who provide a unified treatment of stochastic renewable resource extraction models in a stochastic growth framework.

It differs from their analysis, however, in two respects. First, we abandon all assumptions of concavity of the objective function and rely instead only on its monotonicity and supermodularity and work in a dynamic programming framework. Second, we introduce to the resource and agricultural economics literature a new way of proving convergence to an invariant distribution of the stock of resource, one which needs neither monotonicity of extraction nor compactness of state space (see below).

In discrete-time stochastic groundwater (in general, renewable resource) management problems, the following structural properties, in particular (a),(c), (d) and (e) (in an infinite horizon model) below are of interest.

- **Property** (a): The optimal withdrawal quantity in period t, $w_t^*(x)$, is increasing in x_t, the stock of groundwater at the start of period t,
- **Property** (b): The optimal withdrawal quantity in period t, $w_t^*(x)$ (which is a function of the groundwater stock x at the start of the period), is the maximizer of a concave function of w,
- **Property** (c): $x_t - w_t^*(x)$, denoted the "re-investment function" in the literature, is nondecreasing in x,
- **Property** (d): $w_t^*(x)$ is nondecreasing in t, where the periods are indexed forward in time and
- **Property** (e): The Markov Chain generated by the optimal policy $w_t^*(x)$, $\{X_t\}$, converges to a unique, stationary distribution.

We note first that in the case of groundwater models, many of the results in Mendelssohn and Sobel (1980) are not directly applicable, primarily due to lack of concavity of the net benefit function (we indicate, after each result, if it follows from any existing framework). Further, the question regarding the concavity of the value function has been answered in the negative in Knapp and Olson (1995). Models used in the literature are replete with assumptions regarding smoothness, in particular, (joint) concavity of the objective function and convexity/compactness of the relevant spaces. Such assumptions appear to be an artifact of the sufficient conditions for obtaining the structural properties listed above, rather than arising from any underlying characteristic of the natural-economic system being studied.

Knapp and Olson (Knapp and Olson (1995, 1996)) move away from such smoothness conditions and work in a lattice-theoretic framework, which we also adopt. However, their formulation of the problem side-steps the issue of uncertainty. In their set up, uncertainty (in surface water flows) is resolved *prior* to the farmer making the extraction decision. In such a set up, the decision maker (farmer) can directly control the succeeding period stock, as a result of which they work directly with next periods stock. Thus, proof of monotonicity of next period's stock in current period stock is sufficient for Knapp and Olson to apply theorems on convergence of monotonic Markov Chains.

The model we use, on the other hand, involves uncertainty being resolved *after* the farmer has made extraction decisions, and corresponds better to a real world scenario in a developing nation wherein farmers make decisions on extraction prior to recharge (rainfall) occurring. Here, the inability to directly control the subsequent period stock leads to uncertainty being central to the farmer's extraction decision. In this set up, as we show below, monotonicity of extraction and reinvestment are not identical, even for the model studied in Knapp and Olson (1995, 1996).

Finally, as a direct consequence of the assumptions (referred to earlier) made in prior literature (e.g. Knapp and Olson (1995)), the Markov Chain generated by the optimal policy in those settings is monotone. This, in addition to the assumed compactness of the state space, allows (as in Knapp and Olson (1995, 1996)) a direct application of standard theorems on convergence of monotone Markov Processes (such as those in Stokey and Lucas (1989) and Hopenhayn and Prescott (1992)) to establish convergence to a unique invariant distribution of the resulting Markov Chain. Compactness of the state space (implying boundedness of shocks) is an essential assumption of these methods.

In this setting, we illustrate the use of more probabilistic methods which guarantee the existence of an invariant distribution even in cases where the Markov Chain is *not monotonic*. The assumptions required for the use of this method are more benign than for the methods commonly used in the literature on dynamic economic models. These methods are applicable to a wide variety of natural resource extraction problems and are of independent interest. In addition, we extend our results on global stability to situations of stock-dependent recharge, the first such result (to our knowledge) in this literature.

To summarize, the model set-up here differs from that in Mendelssohn and Sobel (1980) in not assuming concavity of G (and also in not working in a stochastic economic growth framework) and from those in Knapp and Olson (1995, 1996) in terms of the timing of uncertainty (uncertainty resolved after the extraction decision is made) as well as the breadth of cost functions accommodated. Our analysis of the effect of risk on optimal decisions, while simpler than that in Knapp and Olson (1996), is more transparent, in that the objective function with risk is a simple transformation of that without (see pp 15). This provides a link between the with- and without risk scenarios, as well clarifying which assumptions are necessary to obtain our results in each case. In addition, we illustrate the use of more powerful and broadly applicable methods for proving global stability.

Using that, we are able to prove global stability for all the Markov Chains for the dynamic systems studied here, under weaker assumptions than in Mendelssohn and Sobel (1980) and Knapp and Olson (1995, 1996).

We emphasize that many of the generalizations to the standard model of single-user managing a single-cell aquifer proposed here are motivated by the necessity of studying different real world systems of groundwater management, particularly from the Indian sub-continent, as we detail below. We refer the reader to Shah (2007), Shah (2010) and Fishman et al. (2011) for further details regarding the groundwater economy of the Indian sub-continent.

Single-user single-cell models of aquifer management have been known to be particularly weak at modeling groundwater flow complexity and decision interactions between multiple agents, and recent literature has focused on considering these relatively complex modeling scenarios (e.g. Athanassoglou et al. (2012), Brozović et al. (2006, 2010), Chakravorty and Umetsu (2003) and Madani and Dinar (2012)). This is particularly an issue of significance in the case of semi-arid regions of the world, which are the largest users of groundwater. In particular, the Indian sub-continent is a large and hydro-geologically varied landmass, with extensive groundwater withdrawals (e.g. Shah (2007)). Groundwater extraction in these regions is characterized by a unique, but common, set of circumstances: lack of regulation of extraction (e.g. regulations on well spacing, extraction quantity and well licensing are absent), significant constraints to investment in extraction capacity (resulting from the small land holding size–smaller than 1 hectare, on average–and constraints on credit), highly unequal sharing of benefits from extraction, significant variation in type and nature of recharge behavior (see Shah (2010) and references therein).

In general, groundwater reservoirs in semi-arid regions of the Indian sub-continent are characterized by highly variable recharge, climate is highly monsoonal, with a very short and intense rainfall season, and long dry periods during which extraction typically occurs. Finally, extraction (subject to availability and capacity constraints) is largely limited by availability of electric energy while regulation of extraction occurs (de facto) via altered electric supply (see e.g. Shah et al. (2008), Fishman et al. (2011)). These features of the groundwater agriculture system illustrate the importance of greater focus on extraction cost of groundwater, treated simplistically in the economics literature, focused mostly on the larger-scale agriculture in the United States.

These special characteristics have resulted in a sparse yet interesting literature in economics, focused on modeling different aspects of the problem; e.g. investments, in Aggarwal and Narayan (2004) and regulation–in a very simple manner and setting–,in Athanassoglou et al. (2012)). Nonetheless, the basic model of single-user managing a single-cell aquifer has so far not been modified to accommodate the distinct features of groundwater and agricultural systems in semi-arid regions in the developing world. These features merit, we believe, greater focus on both the cost of pumping and on more complex transition equations for groundwater (see also footnote 3). This study is an attempt to integrate some of the most important features of groundwater management in the setting described above into a standard stochastic resource economics framework.

The three relevant dimensions of complexity in groundwater management models relate to treatment of the economic component, the hydrologic component and the treatment of uncertainty. Given the intractability of combining all aspects in a single model, most studies have chosen specific aspects to focus on. Thus, while studies such as Athanassoglou et al. (2012) which model heterogeneous (and Brozović et al. (2010), Madani and Dinar (2012) which model homogenous) users with less restrictive hydrological assumptions advance understanding of the effect of complex interactions on optimal policies, they are are unable to account for the effect of uncertainty (in addition to suffering from some of the drawbacks inherent to complex models such as specific functional forms). Our study, on the other hand, fully accounts for the effect of uncertainty on stock dynamics and, in addition, attempts to improve modeling of certain hydrologic features, while remaining in a single-cell framework. Thus, our analysis may be seen as both complementing those analyses seeking improved modeling of complex hydrologic interactions and extending the current models of single-user groundwater management to more diverse situations.

2 Model Structure

2.1 Setup

We describe a discrete-time stochastic groundwater management model, based on the classical paper by Burt (1964), which is standard in the literature. We consider a finite-horizon problem where T is the planning horizon, and periods are indexed forward by $t \in \{1,2,\dots,T\}$. Let $x_t \geq 0$ denote the groundwater stock[2] at the beginning of period t. The manager decides the withdrawal quantity $w_t \geq 0$. Then a non-negative random variable corresponding to the recharge to groundwater stock, R_t, is realized. We suppose that the $\{R_t\}$ are independent and identically distributed, but indicate generalizations where appropriate and feasible. We assume that the state transition for the groundwater stock level is given by[3]:

$$x_{t+1} = \tilde{X}(x_t, w_t) = x_t - w_t + R_{t+1} \tag{1}$$

[2]Following the literature cited above, we work with the total stock of water instead of the *lift*, which is more conventional in the engineering literature. However, as remarked in Worthington et al. (1985)(pp 235), since lift increases as groundwater decreases, both are related via a monotonic function and therefore, one may work with either, w.l.o.g.

[3]The use of such a simplified balance equation, as remarked in Worthington et al. (1985)(pp 232-233) is a gross over simplification. Taken literally, this equation implies an "instantaneous" capture of all recharge by any pumping activity. However, this oversimplification can be remedied by (i) introducing relevant coefficients on recharge such that only a fraction of the recharge is captured (ii) using more detailed and accurate equations of motion for the stock (where available). Finally, in the case of many developing nations where discharge due to pumping is much larger than recharge, such as assumption is less of an oversimplification, since a large part of recharge is very likely captured within the region of pumping.

Equation (1) corresponds to a reservoir of infinite capacity and is most commonly used. We however illustrate (see remark 4) which of our results survive when a more realistic finite reservoir of groundwater ($\bar{x} < \infty$) is assumed (as in Knapp and Olson (1995)), with state transition

$$x_{t+1} = \tilde{X}(x_t, w_t) = \min(x_t - w_t + R_{t+1}, \bar{x}) \tag{2}$$

The single-period benefit (or reward) is

$$G(x, w) = B(w) - C(x, w) \tag{3}$$

where $B(w)$, concave and increasing, is the net benefit of withdrawing w units of water before deducting the pumping cost $C(x, w)$ (which we address shortly). The objective is to maximize the present value of the benefit

$$\sum_{t=1}^{T} \delta^t G(x_t, w_t) \tag{4}$$

where $\delta \in (0, 1]$ is the discount factor. Then, the value function of dynamic programming is given by the following recursion:

$$V_t(x_t) = \max_{w_t \geq 0} \{G(x_t, w_t) + \delta E[V_{t+1}(x_t - w_t + R_t)]\} \tag{5}$$

where $V_{T+1}(x_{T+1}) = 0$.

We note that, unlike in much of the economics literature on the one-sector stochastic growth model, wherein focus is on the infinite horizon problem, we begin with a finite horizon set up and indicate extensions to an infinite horizon. The reason for this two step approach is the added understanding provided in the finite horizon set up, especially with regard to such intuitive questions as the behavior of the optimal policy with a lengthening horizon. This approach also makes transparent which type of assumptions regarding the state space and the benefit functions maybe relaxed. Finally, finite horizon models are also more commonly used in the resource and agricultural economics literature.

We turn next to characterizing the cost functions we use, and comparing them with those used in the literature.

2.2 Cost Functions

Pumping cost functions used in the literature mostly make the assumption of constant volume pumped as well as of constant (i.e. independent of w_t) marginal cost of pumping i.e. $\dfrac{\partial C(x_t, w_t)}{\partial w_t} = c(x_t)$. Within this broad framework, with the exception of Worthington et al. (1985) and Burness and Brill (2001), most papers (Zeitouni (2004); Aggarwal and Narayan (2004); Tsur and Graham-Tomasi (1991); Rubio and Castro (1996); Roseta-Palma and Xepapadeas (2004); Provencher and Burt (1993)) use a cost function of the form

$$C(x_t, w_t) = c(x_t)w_t \tag{6}$$

with $c(x_t)$ (generally called the marginal cost function) either linearly decreasing in x_t or decreasing and convex, with the most common functions being $c(x_t) = (a - bx_t)$ or $\frac{a}{bx_t}$ (except for Roseta-Palma and Xepapadeas (2004) who use $c(x_t, w_t) = aw_t$, a stock independent linear cost). Burness and Brill (2001) use a slightly different formulation for $c(x)$ while maintaining the separability outlined above. Their marginal cost function (in our terminology) is of the form $c(x) = \frac{c_0(a-x)}{bc(x-d)}$ with $a, c, b, d, c_0 > 0$ and $x > d$. This function also has all the usual properties $(c(x)_x < 0, c(x)_{xx} > 0)$ that the simpler functions above do. Finally, Worthington et al. (1985) use an empirical form of the marginal cost function (also separable), estimated (interpolated) from data. Their function is decreasing but, due to being a combination of different order polynomials for different regions, has $c(x)_{xx} \lesseqgtr 0$ depending upon x.

In general, all cost functions used make the following two assumptions:

1. Marginal cost of pumping does not depend upon the quantity pumped i.e. $\frac{\partial C(x_t, w_t)}{\partial w_t} = c(x)$

2. Cost of pumping does not vary within a given season i.e. drawdown does not vary within a given season. This leads to specifications in which marginal costs are based entirely on beginning-of-period water level depth. This assumption is completely implicit in almost all studies, discrete- and continuous-time, and is made clear only in Worthington et al. (1985) (pp 235-237), wherein (in their Table 1, pp 236) they estimate an *annual incremental drawdown* given only the level of water table at the *beginning* of the season, *independent* of the quantity of water pumped.

The first characteristic is a modeling assumption and is not always in accord with hydrologic facts (see below). The second characteristic above is reasonable if the length of a period is sufficiently small and decisions are made frequently, in which case continuous-time models are more appropriate. In sum, in all of the literature, marginal costs of pumping do not vary within season, and are independent of the quantity pumped during the period. The implications is that, for the marginal cost of pumping in any *given period*, the only factor of importance is the groundwater stock at the beginning of the period; the amount of abstraction during the period does not matter.

With this cost function, structural properties such as the concavity of the optimization problem in each period have not been established [4]. As already indicated, there have been relatively few efforts focused on establishing structural properties of discrete-time groundwater models in the literature and many of the properties still have not been shown to hold, even in a setting with simple (and physically unrealistic) cost functions. In our analysis, we generalize this formulation for cost to reflect more realistic conditions on marginal cost and study in detail the structural properties of the resulting dynamic programs. In particular, we seek to address two important questions:

1. Under what conditions, for the given objective function(s), do intuitive properties (such as monotonicity of withdrawal, next period's stock in current period stock and existence of a stationary distribution for the stochastic stock) hold?

2. Which of these properties hold under a setting where risk aversion is introduced?

We first provide some motivation for the two cost functions we consider below. We have already noted the importance of cost of extraction for determining extraction in the context of agriculture in the Indian subcontinent. We also point out that, in the context of a single user managed resource, the effect of other users' pumping is by definition assumed out. Nonetheless, even with a single user in a more realistic aquifer, two essential features are currently missing from discrete-time models in the literature: 'path dependence' of resource stock and the effect of formation of localized "cones of depression" (Brozović et al. (2006, p. 2-4)).

We carry out two generalizations of the cost function, in light of remarks above. In the first,

$$C(x,w) = \int_{z=x-w}^{x} \gamma(z)dz \tag{7}$$

where γ is assumed to be non negative, decreasing and convex. This form of the cost function takes account of changes in the groundwater stock that occur during the pumping period. Further, the marginal cost of pumping depends, in general, on the quantity of water pumped[5]. We stress that the conventional approach, in eq. (6), with the implication that marginal cost of pumping is independent of the quantity pumped, is an assumption (recognized explicitly in Chakravorty and Umetsu (2003)(pp 5)), and has little empirical or theoretical support.

This cost function imposes a form of path dependency *within* a given season. This is (arguably) quite reasonable in the context considered[6]. We provide two interpretations of this cost function. In the first (com-

[4] Provencher and Burt (1993)[pp 146]show the concavity of the value function, but their result crucially depends on (a) the separability of $C(x,w)$ (b) the fact that the per-unit pumping cost is independent of the quantity, w_t, of groundwater extracted and (c) an unsubstantiated claim of concavity of the objective function. See footnote 11.

[5] Certain other properties of the cost function are enumerated below but intuitively, with this cost function, extraction of the n^{th} unit of water is less costly than extraction of the $(n+m)^{th}$, $\forall m,n>0$, and for all stock levels. In the particular functional form used, the marginal cost of extraction is $\gamma(x-w)$, a function of both x and w, with $\frac{\partial^2 C}{\partial x \partial w} = \gamma' < 0$ and $\frac{\partial^2 C}{\partial w^2} = -\gamma' > 0$.

[6] We note the intimate relationship between model time-step and model formulation. In the context considered here, a time period is a full growing season, approximately of 6 month duration, and in some part (or all) of the growing season, groundwater is the only source

pletely ignoring the effect of other pumping wells upon the well considered), this is the cost function for a user extracting water over a relatively long season, and the extraction (relative to existing stock) is “large”. In the second, one may consider two hypothetical pumping wells, of very similar size, located a small distance apart; note that since each user is homogenous and each well is identical, each user extracts the same amount, w. In this scenario, this is the cost function of either user. Thus, pumping costs for either user are a function only of $x - aw$, where a is a parameter.

The latter interpretation can be justified by an appeal to the the discussion accompanying figure 1.2 in Brozović et al. (2006), which makes clear that for many aquifers, if two pumping wells are only a small distance apart (which is certainly the case for the context considered here) only pumping in the immediate past is of any relevance. Given the definition of a season (about 6 months), and the relatively small distance separating pumping wells in the context considered, our modeling of “path dependence” in pumping as (i) being restricted to a season and (ii) affecting cost but not stock, is quite natural. The lack of an effect upon the stock here is a reflection of the relatively short-term effect of pumping wells upon one another and the length of the season considered.

In the second, we generalize the cost function to accommodate finite aquifer transmissivity, in the context of pumping wells located in close proximity. There has been an increasing body of literature (see for instance Chakravorty and Umetsu (2003); Zeitouni and Dinar (1997); Brozović et al. (2006) among others) dealing with a variety of issues related to managing such common pool resources in realistic hydrologic settings. The main issue involved is the formation of “cones of depression” around individual wells, leading to increasing extraction costs. These costs depend, to a large extent, on a variety of physical parameters such as transmissivity and storativity (see for instance Athanassoglou et al. (2012)).

On the other hand, even in the case of a single user, the assumptions made in almost all models regarding the aquifer are unlikely to hold for any given aquifer. For instance, most continuous time models, following Gisser and Sanchez (1980), assume an essentially bottomless aquifer. Further, most analyses (exceptions include Worthington et al. (1985)) assume implicitly or explicitly an unconfined aquifer. However, for confined aquifers, ongoing pumping from a well induces a localized “cone of depression” around the well. These dynamics lead to increases in the per-unit cost of extraction by increasing the effective lift[7].

We seek a simplified framework to account for the increased marginal cost of pumping as a result of such

of water available. Given the length of the season, it is not clear that cost of extraction ought to depend upon beginning-of-period stock of water, especially if extraction is likely a large part of total stock of water (e.g. in Telangana region, see Fishman et al. (2011)).

[7]This issue, even with the confined aquifer assumption, has been remarked on before. For instance, Provencher and Burt, 1994 (p. 882) clearly recognize that groundwater levels do not adjust immediately after a local perturbation caused by pumping from the well due to the fact that the equilibrating flows are both slow and subject to great variability. In particular, with heterogeneous aquifers and unequal pumping rates by farmers, they clearly recognized a not insubstantial lateral flow of groundwater, leading to uncertainty regarding future water availability. While their discussion was in the context of a common property problem, it is clearly applicable in the context of our modification to the cost function.

localized cones of depression. To understand the approach, consider first a finite difference cell approach to groundwater modeling[8], at a relatively coarse scale (e.g. 25×25 km resolution, commonly used for many developing countries). The drawdown calculated in these models does not represent the actual drawdown around a pumping well. However, if the relevant physical parameters are known, a term linear in extraction, q, derived from the Theis-formula, may be added to correct for this difference between actual and model predicted drawdown; the correction leads to increased pumping costs[9]. In our case, given that there is only one pumping well and that therefore changes in stock due to limited aquifer transmissivity cannot affect subsequent period stock (see discussion in p.14), we incorporate this effect as a linear (in w) term on the cost of extraction i.e. as a change in stock level for purposes of computing pumping cost. Alternatively, this may also be viewed as a first-order approximation to a more complicated (possibly polynomial) correction factor, and at any rate, serves as a approximation to the actual increase in pumping cost. Abstracting away from parametrization, we translate this into our notation as follows:

$$C(x,w) = c\,(\bar{x} - x + aw)\,w \tag{8}$$

where the additive term aw captures the "correction" to the drawdown (change in stock)[10] and, depending on the transition equation, $\bar{x}$ is interpreted as a effective or total stock. Marginal cost, $c\,(\bar{x} - x + 2aw)$, is evidently a function of both x and w, and is based on the beginning of period stock, unlike in eq. (7).

Again, there are two possible interpretations of this cost function. As for the cost function in eq. (7), for a single pumping well scenario, when the season considered is relatively long, accounting for cones of depression via the cost function captures (we believe) the essence of the issue, that of increased cost of pumping. In addition, if seasons are long enough, there are unlikely to be effects upon next-season stock (to a first order) of finite aquifer transmissivity. A second interpretation is also available, for two homogenous users (implying identical extraction) pumping from similarly sized wells. A hydrologic fact, first, is that (to a first order approximation) a well pumping larger quantities has a deeper cone of depression (e.g. Brozović et al. (2006, p.3)). In addition, finite aquifer transmissivity implies a identical reduction in lift in both pumping wells; given that these effects are unlikely to carry over to subsequent season stocks (already discussed above), the only effect of well interference upon each user then can only be modelled as an identical increase in each

[8]The finite difference approach is the standard approach to $2-$ and $3-$dimensional groundwater flow modeling; e.g. the USGS's MODFLOW (Harbaugh et al. (2000)).

[9]The precise correction is given by the formula $0.3665\,\frac{q_{i,j}(t)}{m_{i,j}T_{i,j}}\log\left(\frac{\Delta x}{4.81 r_{BH}}\right)$, where i, j refers to the finite difference cell index, m is the number of pumping wells in the cell with uniform pumping, T is the transmissivity, Δx and r_{BH} are respectively the cell size and radius of the well. See e.g. Siegfried (2004) (pp 52-53) for a description and Prickett and Lonnquist (1971) for the derivation.

[10]For subsequent analysis, we drop the coefficient a on w for minimizing the number of constants used. This coefficient maybe introduced as needed for empirical analysis and its exclusion does not fundamentally impact our analysis but does lead to simpler algebraic manipulations.

user's pumping cost.

This framework (cost in eq. (8) and transition in eq. (1)) is an uneasy combination of a particular form for stock-dependent cost with an infinite depth aquifer. See remark 4 for a generalization. The use of the simplified transition equation (eq. (1)) here is for sake of unification, since our results with the cost function in eq. (7) do not extend to the more realistic transition in eq. (2).

3 Analysis: Finite Horizon

We consider the dynamic model of groundwater extraction using each of the three cost functions, separately, since many features, method of proof and conclusion differ. The approach taken in our analysis is somewhat different from that conventionally used in the resource and agricultural economics (and economic growth) literature. We abandon, for the most part, *all assumptions regarding smoothness* (concavity, convexity and differentiability) of the objective function and feasible set, except insofar as these properties can be *proved* from physically or economically reasonable primitives. We use lattice-theoretic methods, developed in Topkis (1978), with a comprehensive account in Topkis (1998) and the detailed exposition of dynamic programming using lattice-theoretic methods developed in Heyman and Sobel (2003) (see also Amir (2005) for an economics-oriented exposition).

We relegate to an Appendix all standard definitions and notations regarding lattice theory. Note that, for our optimization problems, $W(x)$ will be identified as the "feasible" set of optima i.e. $w_t(x) \in W(x)$, where $w_t(x)$ is extraction at time t, with start of period stock x. Optimal extraction will be indicated by $w_t^*(x)$. Unless otherwise indicated, $w_t(x)$ and $w_t^*(x)$ are (possibly singleton) sets. In order to minimize notation, $w_t^*(x)$ will also be used for the largest element of the set $w_t^*(x)$, where the context should make it clear if $w_t^*(x)$ refers to a set or its largest element. Finally, we use Π and G interchangeably to denote the one-period benefit (profit) function.

Definition 1. (Re-investment function)$a = x - w$ is the "reinvestment function". It will be used in the proof of monotonicity in t of the value function and the optimal policy function (where applicable). It is trivial to reformulate all of the statements in terms of a instead of w, with $A(x)$ the equivalent of $W(x)$ for a.

Assumption 1. $\Pi(x,w)$ *is supermodular in* (x,w) *and increasing in* x

Assumption 2. $\Pi(x,a)$ *is supermodular in* (x,a) *and increasing in x.*

Assumption 3. $\tilde{X}(x_t,w_t)$ *is increasing in* x_t*, given* (w_t,R_t) *(or independent of* x_t*).*

Assumption 4. $\Pi(x,w) \geq 0$, $\forall (x,w) \in \mathscr{C}$

Assumption 5. Π *is finite on* $\mathscr{C}$

Assumption 6. $\exists B > -\infty$ *s.t.* $\Pi(x,w) \geq B$, $\forall x,w$ *i.e.* Π *is bounded below by a number independent of* (x,w).

Assumption 7. $\Pi(x,a) \geq 0$

Assumption 8. $W(x)$ $(A(x))$ *is ascending in* x *on* X.

Assumption 9. $W(x)$ $(A(x))$ *is expanding in* x.

Assumption 10. $W(x)$ $(A(x))$ *is compact.*

Assumption 11. $\tilde{X}(x_t, w_t)$ *is stochastically supermodular.*

Assumption 12. $\tilde{X}(x_t, a_t)$ *is stochastically supermodular*

We briefly discuss the content of these assumptions in the context of our model, relative to those made in the literature. We begin with noting that Assumptions 1 and 2 shall be proved for each model considered, while assumption 6 is clearly a very reasonable practical requirement. Assumptions 4 and 7 clearly follow from 6 (see Remark 2), and assumption 5 is the only restrictive assumption. We note that in the conventional dynamic programming framework, one of the two Assumptions, 5 and 6 (along with a compact $\mathscr{C}$) must hold; these are implicitly or explicitly assumed in the existing literature. Finally, assumption 5, the most restrictive assumption, is only needed for proving that $V_t \to V$, in proposition 1.

Turning now to assumptions regarding $W(x)\,(A(x))$, Assumptions 8 and 9 are clearly reasonable, and are made in all models in the literature; assumption 10 is our only compactness assumption and is needed only to ensure uniqueness of the maximizer. Finally, while assumption 3 holds for both of our state transition equations (eq. (1) and eq. (2)) while assumption 11 (and assumption 12) is a restrictive but common one in dynamic programming with lattice theory (see Heyman and Sobel (2003)[pp 381-3] for a discussion).

For instance, Knapp and Olson (1995, 1996) implicitly or explicitly make Assumptions 2, 3, 7, 5, 8-10 and 12, while Mendelssohn and Sobel (1980) (and its generalizations in Heyman and Sobel (2003)) use, in addition, assumptions regarding concavity of G (or postulate a special form for G); finally, traditional dynamic programming analyses (as in Provencher and Burt (1993)) implicitly use assumption 5 and other compactness and convexity assumptions.

We repeat here the dynamic programming recursion from eq. (5), with

$$V_t(x_t) = \max\{J_t(x_t, w_t); x \in X, w \in W(x)\} \tag{9a}$$

$$J_t(x_t, w_t) = \Pi(x_t, w_t) + \delta \mathbb{E}[V_{t+1}(x_t - w_t + R_t)] \tag{9b}$$

4 Conventional Cost Function (eq. (6))

Recall that the cost function is:

$$C(x,w) = c(x)w = c(\bar{x} - x)\,w \tag{6}$$

where $\bar{x}$ is as in eq. (8).

Thus, the objective function (which we call Π for this section) is:

$$\Pi(x_t, w_t) = B(w_t) - c(\bar{x} - x)\,w \tag{10}$$

Two facts may be inferred from eq. (10). First, Π is not jointly concave in (x,w)[11]. Second, Π is supermodular in (x,w). This may be shown using the differential characterization i.e. a smooth (for instance $\mathscr{C}^2$) real valued function $f(u,v)$ on a lattice is supermodular in (u,v) iff $\dfrac{\partial^2 f(u,v)}{\partial u \partial v} \geq 0$[12]. That this is unconditionally true is evident from the fact that $\dfrac{\partial^2 \Pi(x,w)}{\partial x \partial w} = c > 0$. Further, it is equally evident that $\Pi_x(x,w) = cw > 0$.

We make a further, technical, assumption before we state our main result for this section.

Assumption 13. *$J_t(x_t, .)$ is upper semi-continuous on $W(x_t)$ for each $t \in \{1, 2, \ldots, T\}$ and $V_{T+1}(x) = 0$*

We are now ready to state our main theorem of this section, which is Heyman and Sobel (2003, Corollary 8-5a). We provide a detailed proof in the Appendix for two reasons: many subsequent proofs will be based on, or refer to, this theorem and no explicit proof is provided in Heyman and Sobel (2003).

Theorem 1. *[Heyman and Sobel (2003),Corollary 8-5a]*

Under Assumptions 1,3,4,8,9,11 and 13, $V_t(x)$ (defined in eq. (9a)) is increasing in x and $w_t^*(x) = argmax\{J_t(x_t, w_t); w_t \in W(x)\}$ is ascending in x on $\{x; x \in X, argmax\{J_t(x_t, w_t)\} \neq \varnothing\}$.

Further, under assumption 10, there exists a least and a greatest element in the set $w_t^*(x)$ and these are both increasing in x.

Proof. See Appendix. □

Proposition 1. *Under Assumptions 4 and 5, $V_t(x)$ is decreasing in t, for every $x \in X$*

[11] To see this, note that: $C_x = -c$, $C_{xx} = 0$, which leads to $\Pi_{xw} = -c$, $\Pi_{xx} = -C_{xx} = 0$, $\Pi_{ww} = -B''$. For Π to be jointly concave in (x,w), the Hessian is required to be positive semi-definite, i.e. $B'' C_{xx} \geq (C_x)^2$ which in this case is equivalent to $c^2 \leq 0$, which is evidently false. Thus, Π does not satisfy a sufficient condition to be jointly concave. Of course, this condition is not necessary for concavity, but it is unlikely that Π will be jointly concave, since Π is otherwise very smooth.

[12] Note that using the differential characterization of supermodularity here does not need special assumptions regarding differentiability. First, in our case, since $B(.)$ and $C(x,w)$ are each $\mathscr{C}^2$, so is Π. Second, it is straightforward to show supermodularity from more primitive conditions, as below. Let $y = (x,w)$, $y' = (x',w')$, with $x > x'$ and $w < w'$; then $y \wedge y' = \min\{y,y'\} = (x',w)$ and $y \vee y' = \max\{y,y'\} = (x,w')$. Π is then supermodular iff $\Pi(y \vee y') + \Pi(y \wedge y') \geq \Pi(y) + \Pi(y')$. It is easy to see that this follows from $c(x)$ being decreasing in x. We note, however, that proving log supermodularity, in section 4.1, in particular in claim 1, is challenging without a $\mathscr{C}^2$ characterization.

Proof. We provide a proof by induction. $V_T = \max_{w_t \in W(x)} (\Pi(x,w)) \geq 0 = V_{T+1}$. Let the hypothesis hold for some $k < T$ i.e. $V_k(x) \geq V_{k+1}(x)$. Consider now

$$\begin{aligned} V_{k-1} &= \Pi(x,w) + \delta \mathbb{E}[V_k(x - w + R)] \\ &\geq \Pi(x,w) + \delta \mathbb{E}[V_{k+1}(x - w + R)] \\ &= V_k \end{aligned}$$

□

4.1 *The effect of Risk Aversion*

We turn now to understanding the implications of moving away from a risk neutral setting (implicitly assumed thus far) to a setting where risk does play a role. To be precise, we now consider an economic agent (farmer) whose objective is to maximize expected utility from profit i.e. $U(\Pi(x,w))$ where U is a strictly increasing and concave function. We note that using relatively realistic objective functions and in particular, accounting for possible risk aversion of decision makers in dynamic decision models has been stressed before in the agricultural economics literature (Krautkraemer et al. (1992) and Kennedy et al. (1994) among others). Nonetheless, apart from Knapp and Olson (1996), there has been no analysis of the structural properties of dynamic groundwater management models in a non-risk-neutral setting and the practical implications of risk aversion for management of groundwater.

Knapp and Olson (1996) consider the problem in a recursive utility framework and show that optimal policies vary significantly when compared to a risk neutral setting. However, the analysis there is limited in that; (i) only Properties (a) and (e) are proved (ii) despite being an extension of Knapp and Olson (1995), there is no explicit link provided to the analysis therein. In other words, important properties in Knapp and Olson (1996) are proved more as a result of certain assumptions regarding model structure than as an extension of the model in Knapp and Olson (1995). To be more explicit, properties (a) and (e) are proved via assumptions on the objective function Π–which is recursive–rather than by considering a recursive utility function U operating on the profit function Π, the latter being defined as in Knapp and Olson (1995). Thus, it is not possible to trace a direct link between the analysis in Knapp and Olson (1995) and Knapp and Olson (1996).

We work instead in a simpler setting using a conventional utility function, the popular log utility. However, we treat the resultant model explicitly as an extension of our analysis for the risk neutral setting and are able to provide a link between both settings. Finally, we also show (see remark 3 below) that the conditions we impose on the utility function, in the risk averse case, are very closely related to those assumed in Knapp and Olson

(1996).

While it is evident that the maximizer cannot change by means of this alteration in the objective function, new difficulties arise as a result of this operation. A major cause for the difficulties encountered is that supermodularity, unlike concavity, is a *cardinal* property i.e. $g(f)$, with g strictly increasing and f supermodular, is guaranteed to be supermodular only if g is also convex. Further, an ordinal generalization of supermodularity, quasi-supermodularity (developed in Milgrom and Shannon (1994)) does not lead to any operational benefits in most cases, since other than strictly increasing transformations of supermodular functions, it is difficult to verify quasi-supermodularity of a function in practice (see e.g. Topkis (1998, pp 60-61)).

It turns out, however, that when restricted to the log function (i.e. when $U(\Pi) := \log(\Pi)$), some pleasing properties are retained. In particular, the following is true:

Claim 1. $\Pi(x,w)$ *is log super-modular (log s.p.m)*

Proof. A function f is log s.p.m if $\log f$ is s.p.m i.e. if $\dfrac{\partial^2 f}{\partial u \partial v} \geq 0$ (Topkis, 1998, pp 64)which implies $f f_{uv} \geq f_u f_v$. In our case, the verification exercise involves $\Pi\Pi_{xw} \geq \Pi_x \Pi_w$, with $\Pi_x = cw$, $\Pi_w = B^{'} - c(\bar{x} - x)$, $\Pi_{xw} = c$. The condition to be verified yields $c\left(B - B^{'}w\right) \geq 0$, which holds only if $B \geq B^{'}w$. That this condition holds for (i) $B(w) = aw - bw^2$ (ii) aw (iii) $ln(w + D)$, $D = 1$ and (iv) $B(w) = 1 - \exp(-aw)$ is easily seen[13]. These four benefit functions more than span the range of the empirical functions used in the groundwater management literature (the most popular of which is that in (i)). □

Replacing $\Pi(x,w)$ in eq. (9b) with $\log(\Pi(x,w))$, noting that $\dfrac{\partial U(x,w)}{\partial x} = \dfrac{\Pi_x}{\Pi} \geq 0$ (whenever Π is bounded away from 0), it is evident that theorem 1 is applicable, which is the content of the following proposition:

Proposition 2. *theorem 1 is applicable if assumption 1 is replaced with claim 1, and assumption 4 with assumption 6.*

Remark 1. (Finiteness of J_t) We comment now on the assumption of boundedness and non-negativity of Π. Note first that, for the purposes of theorem 1, in particular, for the maximization operation, it is required that either Π be finite or that $\mathscr{C} = \{(x,w); w \in W(x)\}$ is compact and $J_t(x,w)$ is upper semi-continuous (both conditions result in finiteness of J_t and the existence of a maximizer), the latter of which we assume in theorem 1. For these purposes, it is *not sufficient* that Π is non-negative, an approach which is most common when applying conventional approaches to dynamic programming for unbounded rewards. ◊

[13] Consider first the function $\log(w + D)$, $D = 1$. For $B > B^{'}w$ for this function, it is required that $\log(1 + w) \geq \dfrac{w}{1+w}$. That this holds is evident from the logarithmic inequality $\dfrac{w}{1+w} \leq \log(1+w) \leq w$. For the function $B(w) = 1 - \exp(-aw)$, it is seen that $B - wB^{'} = 1 - \exp(-aw)(1 + aw)$. That this is non-negative is evident from the following inequality: $e^{-x}(1+x) < 1$, with $x = aw$.

Remark 2. (Boundedness of Π) For the particular case of the logarithmic utility function (unbounded both above and below), it is not sufficient that X is compact, since $\Pi = 0$ is always a possibility. However, this can be easily dealt with in a very general manner, as follows: let Π be bounded below (when Π is finite or defined over a compact set, this is trivial) by $B > -\infty$. Consider now the function $\tilde{\Pi} = \Pi + (1 + B)$. It is evident that $\tilde{\Pi} > 0$ and further, that replacing Π with $\tilde{\Pi}$ yields the same optimal decision. Therefore, there is no loss of generality in assuming Π to be bounded away from 0 in both verification of log supermodularity as well as in claim 1. ♢

Remark 3. (Log supermodularity) Knapp and Olson (1996) consider an identical problem to that in Knapp and Olson (1995), using a recursive utility framework. Their main assumption regarding the function Π (Knapp and Olson, 1996, A.4., pp 1007) is

$$\sigma \Pi \Pi_{xw} \geq \Pi_x \Pi_w \tag{11}$$

It is obvious that, for $\sigma \geq 1$, this condition is implied by the log supermodularity of Π. On the other hand, they use a value of $\sigma < 1$, for which this condition, in fact, is more demanding. Using the integral of the linear demand function used in Knapp and Olson (1995) i.e. using $B(w) = aw - bw^2$, and the standard cost function, $C(x,w) = c(x)w = c(\bar{x} - x)w$, for the condition in eq. (11) to hold, it is necessary that $a \geq \left(\frac{\sigma - 2}{\sigma - 1}\right) bw$. That this condition is stronger than that implied by $B_w(w) \geq 0$ ($\implies a \geq 2bw$) for any $\sigma < 1$ is evident. They appear not to recognize the condition in eq. (11) as a (stronger) form of log supermodularity. ♢

5 Cost function accounting for local cones of depression

We begin first with the risk neutral setting and prove that the use of a seemingly more complicated cost function does not complicate the analysis. Indeed, there is no change in the structural results obtained above.

Recall from eq. (8) that the cost function is

$$C(x,w) = c(\bar{x} - x + w)\, w \tag{12}$$

while the objective function is

$$\Pi(x_t, w_t) = B(w_t) - c(\bar{x} - x + w)\, w \tag{13}$$

From $\frac{\partial^2 \Pi}{\partial x \partial w} = c > 0$, it is evident that Π is supermodular, and that $\Pi_x = cw \geq 0$. This leads immediately to the following propostion:

Proposition 3. *For the net benefit function in eq. (13), theorem 1 is directly applicable.*

Further, if we assume finiteness of Π, then it is straightforward that proposition 1 is directly applicable.

Proposition 4. *For the net benefit function in eq. (13), proposition 1 is directly applicable.*

We turn next to characterizing the properties of the decision problem when one uses a (strictly) concave transformation of net benefits. The issues confronted with this seemingly negligible change are substantial, as discussed in section 4.1. However, even with this cost function, we may show that the profit function is log supermodular, which is the content of the following Claim.

Claim 2. Π *in eq. (13) is log supermodular, under assumption 6.*

Proof. Consider $\Pi_w = B^{'} - c\left(\bar{x} - x + 2w\right)$, $\Pi_x = cw$, $\Pi_{wx} = c$, which yields

$$\begin{aligned}\Pi\Pi_{xw} - \Pi_x\Pi_w &= c\left(B - wB^{'}\right) - c^2 w\left(-w\right) \\ &= c\left(B - wB^{'}\right) + c^2 w^2 \\ &\geq 0 \text{ if } B > wB^{'}\end{aligned}$$

which holds for the four functions for B exhibited in the proof of claim 1. □

This leads to the following proposition:

Proposition 5. *theorem 1 is applicable if assumption 1 is replaced with claim 2, and assumption 4 with assumption 6.*

Remark 4. Observe that the transition function in eq. (2) (in common with that in eq. (1)) is increasing in x i.e. for $x \leq x'$, $\min\left(x - w + R, \bar{x}\right) \leq \min\left(x' - w + R, \bar{x}\right)$. Noting that the only property of the transition function used in the proof of theorem 1 is precisely this, of increasing in stock x, it is evident that all of the results in the preceding two sections, using the cost functions in eq. (8) and eq. (6), hold for the more realistic, finite aquifer case represented by the transition in eq. (2). ◊

6 Accounting for impact of pumping on marginal cost

We show here that all of Properties (a)-(d) hold for the model in eq. (7). This is in contrast to the models in eq. (6) and eq. (8), for which we are able to show that only Property (b) holds. In a related model, Knapp and Olson (1995) are able to show that only (a) holds – and that with the help of an additional condition on c. The underlying *assumption* of Knapp and Olson (1995) is the supermodularity of $B(x-y) - C(x, x-y)$ in (x, y), where $B(w)$ is the benefit of withdrawing w units of water, but this condition sometimes fails for the model in eq. (6); however, for eq. (7), this supermodularity condition always holds, thereby enabling Property (c) to hold.

A common way to prove results such as (a)-(d) is via a dynamic programming analysis that simultaneously proves a result about the concavity of the value function. We use this approach, with the slight variation that it is the concavity of the value function *plus* an expression involving the current state that is used in the analysis. We note that we only rely on Assumptions 3, 5, 8, 9 and 10 for proving our main result, theorem 2. In particular, while Π turns out to be supermodular, we do not make use of it in our proof below.

Due to the fact that the main results here, theorem 2 and corollary 1, have already been published in Huh et al. (2011), we provide only the statement of the main results, in addition to some intuition for these results. These results are included here for two reasons. First, they illustrate the breadth of properties obtainable using methods similar to those used for the other two cost functions, and complement the other results. Second, subsequent sections extend the results in this section (finite horizon) to an infinite horizon and characterize the invariant distribution for this cost function; results from the current section are an integral part of, and a key to understanding, these extensions. We also refer the reader to this paper for a numerical example illustrating the failure of Property (c) and Property (a), with the cost function in eq. (7).

6.1 Main Results

Recall from eq. (7) our cost function $C(x,w) = \int_{z=x-w}^{x} \gamma(z)dz$. From the definitions of V_t and G, we can write $V_t(x_t) = \max_{w_t \geq 0} U_t(x_t, w_t)$, where

$$U_t(x_t, w_t) \quad = \quad B(w_t) - \int_{z=x_t-w_t}^{x_t} \gamma(z)dz + \delta E\left[V_{t+1}(x_t - w_t + R_t)\right] . \tag{14}$$

For any x_t, let $w_t^*(x_t) = \arg\max_{w_t \geq 0} U_t(x_t, w_t)$.

In theorem 2 below, part (ii) shows that the problem facing the decision maker in each period is the maximization of a concave function, which is Property (b). Part (iii) establishes two properties of the optimal decision in each period – that the optimal withdrawal quantity increases in the groundwater stock, and that the groundwater stock in the next period is increasing in the groundwater stock in the current period – which are Properties (a) and (c), respectively. We prove these results by showing that a modification of V_t exhibits the concavity property – which is the content of part (iv).

Theorem 2. *[Huh et al. (2011, Theorem 1)]*

(i) $V_t(x_t)$ is increasing in x_t for each $t \in \{1, \ldots, T+1\}$,

(ii) $U_t(x_t, w_t)$ is concave in w_t for any x_t for each $t \in \{1, \ldots, T\}$,

(iii) w_t^* *satisfies* $w_t^*(x_t) \leq w_t^*(x_t+\varepsilon) \leq w_t^*(x_t)+\varepsilon$ *for any* x_t *and* $\varepsilon > 0$ *for each* $t \in \{1,\dots,T\}$*,and*

(iv) $V_t(x_t) + \int_{z=0}^{x_t} \gamma(z)dz$ *is concave in* x_t *for each* $t \in \{1,\dots,T+1\}$.

The following result shows that with more periods to go, the optimal decision is more conservative in the extraction of water (Property (d)), and implies in particular that the optimal extraction quantity is bounded above by the myopic withdrawal quantity (which corresponds to the last period withdrawal).

Corollary 1. *(Huh et al., 2011, Corollary 1)* $V_t(x)$ *is submodular in* (t,x)*. Furthermore, for any* x *and* $t \leq T$, $w_t^*(x) \leq w_{t+1}^*(x)$.

Remark 5. (Log supermodularity) We remark that the profit function associated with this problem (eq. (13)) is not log supermodular, as a result of which, there is no result analogous to proposition 5. ♢

Remark 6. We remark that, from theorem 2(iii), it is evident that $\frac{\partial w_t^*(x_t)}{\partial x_t} \in [0,1]$, if it exists. This derivative exists almost everywhere since w_t is bounded, increasing and continuous. This result is analagous to Corollary to Proposition 1 in Knapp and Olson (1995) ♢

Remark 7. So far, we have not touched upon an interesting result in Knapp and Olson (1995), Corollary to Proposition 1 (except for the case of the cost function in eq. (7) in remark 6 above). It is important to note, that this Corollary is essentially a result regarding the Lipschitz continuity of $w(x)$, and requires that X_{t+1} is increasing in X_t. In our set-up here, except for the cost function in eq. (7), it is not the case that X_{t+1} is increasing in X_t. Nonetheless, we indicate that under relatively mild assumptions on w', it is possible to establish a result similar to this Corollary for the other two cost functions (in eq. (6) and eq. (8)).

Consider the following facts: $w^*(x)$ (or the largest element of $W(x)$) is monotonic in x, implying that it is a.e. differentiable; making, in addition, the (relatively mild) assumption that the derivative is *continuous*, it is immediate that $w^*(x)$ is locally Lipschitz[14]. Given the intuitive property that beyond a certain point we do not anticipate further increases in X to lead to increased rate of pumping, we already anticipate that $\lim_{x\to\infty} w^{*'}(x)$ is finite, say $0 < C_1 < \infty$; if in addition we introduce the additional assumption that $\lim_{x\to 0} w^{*'}(x) = C_2 < \infty$, claim 3 follows[15, 16]:

Claim 3. $w^*(x)$ *is globally Lipschitz continuous.*

[14]A function f is *locally* Lipschitz if, for $\forall x_0 \in X$, $\exists r > 0$ such that f is Lipschitz continuous on $B_r(x_0)$, an open ball centered at x_0, with constant $\Lambda(x_0)$ i.e. if $|f(z) - f(x_0)| \leq \Lambda(x_0)\ \forall z \in B_r(x_0)$. If $\exists$r for which the same Lipschitz constant Λ applies $\forall x_0 \in X$ then f is said to be *globally* Lipschitz.

[15]In other words, it is very reasonable that w is concave in x, with bounded derivative on $[0,\infty)$. However, while a concave w can certainly be made (by suitable alterations) to satsify the above two conditions (e.g. by assuming $0 \leq C_2 < C_1 < \infty$), concavity is *not* necessary for these conditions. In other words, we do not assume that w is concave.

[16] claim 3 follows from standard results indicating that (i) every continously differentiable function is locally lipschitz and (ii) if in addition the derivative is uniformly bounded (i.e. bounded by a number independent of x) then the function is globally Lipschitz. By the two assumptions above, it is evident that $0 < K := \sup\left\{w'(x); x \geq 0\right\} < \infty$ and therefore, K may be used as the Lipschitz constant.

The following result is therefore immediate:

Corollary 2. *For $x > x^{'} > 0$ denoting two different stock levels and w, $w^{'}$ denoting corresponding optimal withdrawals, it is true that $w^*(x) - w^*\left(x^{'}\right) \leq C\left(x - x^{'}\right)$, for some $0 < C < \infty$.*

We observe that corollary 2 holds for the more realistic transition functions in eq. (20) and eq. (19). While corollary 2 requires assumptions regarding $w^{*'}$, we observe that these are weaker than assumptions regarding differentiability of the value function which are common in the literature using continuous time models (e.g. the existence of the third derivative of the value function in Tsur and Graham-Tomasi (1991)). ♢

7 Effects of the time horizon

We have already proved the following properties regarding the effect of time horizon: for the cost functions in eq. (6) and eq. (8), V_t is decreasing in t (and finite, for each t) in Propositions 1 and 4; for the cost function in eq. (7), it was proved, in addition, that the optimal policy function $w_t(x)$ is decreasing in t, in Corollary 1.

A natural next step therefore is to ask the following questions regarding the limit functions. To fix matters, consider the following two equations, the infinite horizon analogues of eq. (9a) and eq. (9b), which may or may not be well defined at this stage:

$$V(x) = \max\{J(x,w); x \in X, w \in W(x)\} \tag{15a}$$

$$J(x,w) = \Pi(x,w) + \delta\mathbb{E}[V(x-w+R)] \tag{15b}$$

1. Does V_t converge?
   (a) If so, does it converge to the Bellman equation, eq. (15a)?
2. Does the optimal policy function $w_t^*(x)$ converge?
   (a) If so, does the limit function inherit monotonicity?
   (b) Finally, does the limit function maximize the right hand side of eq. (15b)?

We answer each of them in turn. We begin with a series of brief remarks on convergence of the value function and the one-period return function, as a prelude to answering the questions posed above.

Remark 8. We have proved that V_t, for all three cost functions, is decreasing in t, increasing x, and is finite (bounded). It is therefore immediate that $\exists V$ s.t. $V_t \downarrow V$ and further, that the limit function V is increasing in x.

Remark 9. It is evident from eq. (9b) that if $V_{t+1} \leq V_t$ then so too is J_t i.e. $J_{t+1} \leq J_t$ (as may be proved by an easy induction on t). Further, either due to finiteness of Π or assumption 13 and compactness of $\mathscr{C}$, it is evident that J_t is bounded over $\mathscr{C}$. Therefore, $\{J_t\}$ is a monotone, decreasing and bounded sequence. It therefore follows that $\exists J$ s.t. $J_t \downarrow J$ for each $(x,w) \in \mathscr{C}$. $\Diamond$

Remark 10. We have already shown, in claim 1 and claim 2, that the profit functions in eq. (10) and eq. (13) are log supermodular. Remarks 8 and 9 are therefore directly applicable to these formulations. $\Diamond$

We make two final, technical, assumptions before we embark on our major result for this section.

Assumption 14. *$J_t(x,.)$ is continuous in w on $W(x)$.*

Assumption 15. *$\mathscr{C} = \{(x,w); x \in X, w \in W(x)\}$ is a sub-lattice of $\mathbb{R}^2_+$.*

We now state our main result for this section, which is Heyman and Sobel (2003)[Thm 8-16]. We provide (in an Appendix) an outline of the proof in order to aid the reader's understanding, since Heyman and Sobel (2003) do not provide an explicit proof.

Theorem 3. *(Heyman and Sobel (2003)[Thm 8-16]) Under Assumptions 1, 6, 8, 9, 10, 11, 14 and 15, $V(x)$ (defined in eq. (15a)) is non-decreasing in x and $\exists w(x)$ non-decreasing in x which satisfies $V(x) = J(x, w(x))$ for $x \in X$.*

Proof. See Appendix. □

Remark 11. theorem 3 addresses question 1. We note that nothing yet has been said about convergence of the optimal policy i.e. regarding $w^*_t(x) \to w^*(x)$ where w^* is presumably increasing in x and is the maximizer of the right hand side of eq. (15b). We address in turn questions 2, 2a and 2b. To begin addressing question 2a, we observe that if the limit function w^* exists, it must be increasing in x. The question thus to be addressed is 2. In the case of the cost function in eq. (7), we have already proved that w^*_t is decreasing in t. Thus, using the upper bound $\bar{X}$ for x, we have that $\{w^*_t\}$ is a decreasing, bounded sequence, which must converge. For the remaining two cost functions, we have been unable to prove that w^*_t is decreasing in t, which is a sufficient condition for convergence. Thus, convergence is not assured for the remaining forms of the cost function, *including the conventional cost function*.

Finally, for the cost function in eq. (7), we turn to see if question 2b can be answered in the affirmative. The main issue is whether the limit function is identical with the maximizer identified in theorem 3. The only sufficient condition for this (Heyman and Sobel (2003)(Thm 8-15)) relies on the concavity of the value function and is inapplicable in our case. We have been unable to derive a sufficient condition or to provide a direct proof, that the limiting function (to which the optimal policy function converges), in the case of cost function in eq. (7), solves the infinite horizon problem in equation eq. (15b). $\Diamond$

8 Stationary Distribution of Stock

We turn now to understanding the conditions under which the Markov Chain generated by the dynamic decision problem for each of the three cost functions (in eq. (6), eq. (7) and eq. (8)) converges to an invariant distribution. Most analyses on establishing convergence to a unique invariant distribution (i.e. "global stability") in renewable resource management (including Knapp and Olson (1996, 1995)) rely on the monotonicity of the "reinvestment function", following the economic dynamics setting in Hopenhayn and Prescott (1992) and Mendelssohn and Sobel (1980). This approach has two major drawbacks: (a) assumptions regarding compactness of the state space are needed, leading to bounded shocks (an undesirable artifact in many applications) (b) these are very difficult to generalize to non-monotonic systems.

There is an alternative approach, popularized in economics in Stachurski (2009), which uses a combination of the function analytic and probabilistic approaches (drawing on the fundamental work in Meyn and Tweedie (1993)), overcoming both drawbacks mentioned above. This method is particularly suited for an analysis of stability for the cost functions in equations (8) and (6).

We illustrate here the use of a very powerful theorem, applicable to *both monotonic and non-monotoninc* Markov Chains, under a set of mild assumptions which are likely satisfied in a variety of natural resource extraction settings. This approach offers two major advantages over more conventional methods alluded to above[17]: (i) it allows the researcher to look beyond monotonic systems, which in many cases in resource economics are an artifact of model assumptions rather than any underlying feature of the natural (or economic) system being studied and (ii) releases the researcher from the strait-jacket of compact-state-space conditions typically imposed on stochastic renewable resource models. Since the use of this approach in economics is relatively recent, and since this approach has not been used (to our knowledge) in resource and agricultural economics settings, we provide a more detailed outline of the method of verifying the conditions sufficient for its applicability.

8.1 The Setup

The generic transition equation, which (following Stachurski (2009)) we label the "Stochastic Recursive System" (S.R.S, henceforth), is

$$X_{t+1} = f(x_t) + R_{t+1} = F(x_t, R_{t+1}) \tag{16}$$

[17]While powerful, the method used here suffers from a major drawback, relative to those used in Mendelssohn and Sobel (1980) and Hopenhayn and Prescott (1992), which is the need for finding a function v which allows a verification of the condition of *drift to a small set* (in Definition 2 below). The function v *depends* on the functional form of $F(.)$ in eq. (17); it is therefore not possible to provide generic conditions for convergence for arbitrary functional forms of F. Nonetheless, for certain functional forms of stock growth in renewable resource economics (such as linear, as here, or power functions), it is typically possible to find a function v for which convergence holds.

where, for this section, we will assume $R_t \sim^{iid} \Phi$, with Φ a continuous distribution assigning strictly positive probability to every subset of $\mathbb{R}_+$. For the S.R.S. in eq. (16), we denote by M the Markov Operator associated with the stochastic kernel, P, whose definitions, along with standard notations and definitions regarding convergence of Markov Processes, are relegated to an Appendix. We only define below two notions which are directly used in the proof of the main result for this section, proposition 6.

Definition 2. (Drift to small set) The kernel P, associated with the operator M, satisfies *drift to a small set* if $\exists v \geq 1$, $v : X \to \mathbb{R}_+$, $\alpha \in [0,1)$ and $\beta \in \mathbb{R}_+$ s.t

$$Mv(x) \leq \alpha v(x) + \beta$$

and all sub-level sets of v are "small" (see definition 12 in the Appendix).

Definition 3. (Global Stability) Viewing $(\mathscr{P}(X), M)$ as a dynamical system, *global stability* corresponds to the existence of a unique fixed point of the dynamical system.

8.2 Main Results

In the case of our dynamical system, the transition equation maybe written

$$X_{t+1} = x_t - w_t + R_{t+1}$$

Consider now a (stationary) policy $w(x_t)$, possibly sub-optimal and non-monotonic in x_t. Under this policy, the Markov Chain that results maybe written as

$$X_{t+1} = x_t - w(x_t) + R_{t+1} = a(x_t) + R_{t+1} = F(x_t, R_{t+1}) \tag{17}$$

The fundamental questions concern (a) existence of atleast one invariant distribution to the Markov chain generated by the policy $w(x_t)$ and (b) uniqueness of the invariant distribution. In the case of the cost functions in equations (8) and (6), the optimal policy $w(x_t)$ is increasing in x_t but it is *not* the case that $a(x) = x - w(x)$ is also increasing. Thus, the S.R.S $F(x_t, R_{t+1})$ in eq. (17) is not increasing in x on X and results on stability of monotone Markov Chains are not applicable. We indicate, following Stachurski (2009), a constructive method of proof which relies on far simpler assumptions than those for monotone Markov Chains, dispensing in particular with the assumption of compactness of X i.e. that $X = \{x; a \leq x \leq b,\ a, b \in \mathbb{R}_+\}$. We first state an important theorem and then indicate how the conditions required here are satisfied in the case of the S.R.S in eq. (17).

Theorem 4. *(Stachurski, 2009, Thm 11.3.36) If the Stochastic Kernel P is aperiodic, irreducible and satisfies drift to a small set, then the system $(\mathscr{P}(X), M)$ is globablly stable with a unique stationary distribution $\Psi^* \in \mathscr{P}(X)$.*

The verification of the conditions of theorem 4 crucially depends on the following assumption:

Assumption 16. *The function $a(x)$ in equation 17 is continuous, satisfies*

$$a(x) \leq \alpha x + c \tag{18}$$

with $\alpha \in [0,1)$, $c \in \mathbb{R}_+$, $R_t \sim \Phi$, ϕ a density which is strictly positive on $\mathbb{R}_+$ and $\mathbb{E}(R_1) < \infty$.

We now indicate why this assumption is reasonable in the context of our set up. First, note that $0 < w(x_t) \leq x_t$ is necessary[18] for assumption 16 to be satisfied for $c = 0$ (with which we work, since it is not critical that c be positive). Thus, it is *required* that $w(x_t)$ is bounded away from 0 for *all positive values* of x, which of course is a reasonable assumption for any reasonably shallow aquifer. In other words, any agent who has incurred the (not insubstantial) fixed costs of accessing the resource (in the case of groundwater, pump and plumbing; in the case of fishery, capital equipment in the form of boats, nets etc) is unlikely to extract 0 quantity.

Conditions similar to assumption 16 on stock, rather than extraction, are often imposed in models of extinction of natural resources (for instance Olson and Roy (2000)). assumption 16 is *not equivalent* to stating that the stock of resources is bounded away from 0 *almost surely* (pp 194). In other words, the assumption above does not require that the stock be strictly positive, a far stronger condition. Rather, the assumption indicates that even at very low levels of stock, it is always optimal to extract a non-zero quantity of water and further, that this quantity is bounded below. Such will always be the case if costs are not "too convex", relative to benefits. For instance, if costs are less convex than benefits are concave, it is very reasonable to assume that extraction will always be positive. Given that Inada-like conditions to ensure interior solutions cannot be used in this setting, this is an assumption, albeit a reasonable one[19].

We now provide three possible approaches to setting the value of the coefficient α, which is the essence of the verification of the conditions in assumption 16.

- For a differentiable a, in fact, assumption 16 is a condition on the derivative of a i.e. $a^{'} \leq \alpha$. For the cost function in eq. (7), we have already shown that $\frac{\partial w(x)}{\partial x} \in [0,1]$. If we make the (mild) assumption that

[18] It is not difficult to see that it is *not sufficient*: observe that what is required is $w(x) \geq x(1-\alpha)$, which is not guaranteed by $w(x) > 0, \forall x > 0$.

[19] Imposing constraints on controls, rather than stocks, is an unusual approach to optimal control. However, conceptually at least, one may think of this constraint as a "penalty" on lack of water. For instance, if in the absence of extracted groundwater, an alternative source of water has to be found, then the costs of extraction has to be unrealistically high for no extraction to occur at all.

$\dfrac{\partial w(x)}{\partial x}$ is bounded below by a number independent of x, i.e. that $1 \geq \beta = \inf\left\{\dfrac{\partial w(x)}{\partial x};\ x \in X\right\} > 0$ then we can set $\alpha = 1 - \beta < 1$.

- When $w(x_t)$ is not differentiable, there are two possible alternatives for α

  1. if $1 > K = sup\{x - w(x); x \in X\}$ then we can set $\alpha = K$.
  2. Another alternative is to consider $\inf\left\{\dfrac{w_t(x)}{x}; x > 0\right\} = \kappa$; if $\kappa > 0$, we can set $\alpha = 1 - \kappa < 1$.

For the developments below, we assume one of the above is true. Thus, for all the models considered here, we henceforth make Assumption 16. We now indicate the chain of reasoning verifying all the properties required of P in theorem 4.

From Stachurski (2009)(pp 293), irreducibility of P follows, while it can be shown easily that every compact subset of X is small for P, from which the aperiodicity of P follows (Stachurski (2009, pp 292)). Finally, finding a function v satisfying definition 2 will suffice to prove drift to a small set, a task taken up next.

Claim 4. *For the function $v = x$, the Markov Operator M associated with the S.R.S in eq. (17) satisfies the conditions (set out in definition 2) for drift to a small set.*

Proof. Using the definition of a Markov operator (see Appendix) and F from eq. (17), $Mv(x) := \int v[F(x,z)]\,\Phi(dz)$, with $F(x,z) = a(x) + z$, we have

$$Mv(x) = \int (a(x) + z)\,\Phi(dz) \leq \alpha x + \beta = \alpha v(x) + \beta$$

where $\beta := \int z\phi(z)dz < \infty$ and the inequality follows from assumption 16 (with α taking any of the three values above). It is immediate that all sublevel sets of v i.e. set of the form $\{x \in X; v(x) \leq K\}\}$, $K \in \mathbb{R}_+$, are compact. □

Remark 12. An alternate proof exists when the kernel P has a density, ϕ (which exists in this case, see Stachurski (2009, Thm 8.1.3)), of the form $P(x,y) = \phi(y - a(x)) > 0$. Using $Mv(x) = \int v(s)P(x,ds)$, and change of variable, $z = y - a(x)$, $v = y$, we have $\int y\phi(y - a(x))dy = \int (a(x) + z)\,\phi(z)dz \leq \alpha x + \beta$, where $\beta := \int z\phi(z)dz < \infty$. ◊

Thus, all the properties required of P in theorem 4 are satisfied, which leads to the main result of this section:

Proposition 6. *For the groundwater models defined by cost functions, in equations (7), (8) and (6), the stock of groundwater converges to a unique, invariant distribution Ψ^*.*

We stress that this result is very general in that it depends only on three assumptions:

i The functional form for F in eq. (17) is of the form $x - w(x)$, irrespective of the form of $w(.)$ (i.e. $w(.)$ need not be monotonic or of a particular form)

ii a stationary policy exists and

iii under a stationary policy, extraction (not necessarily monotonic) is bounded away from 0.

(ii) is always true while (iii) is clearly an assumption and, we argue, a very reasonable one in many renewable resource settings. This result is thus applicable to any renewable resource setting which satisfies the three conditions and is quite a general result.

Remark 13. An immediate question to be dealt with is the applicability of proposition 6 to the more realistic S.R.S in eq. (20). It is evident, from

$$\min\left(a(x)+R,\bar{x}\right) \leq a(x)+R,\ \bar{x}>0, R>0 \text{ given}$$

that claim 4, with $v = x$, is directly applicable. However, proving aperiodicity and irreducibility in this case is a significant challenge, especially since for the two profit functions for which this transition equation is applicable, eq. (10) and eq. (13), the resulting reinvestment (and therefore the S.R.S) is non-monotone. However, via an alternate route, and using the compactness of the state space ($X = [0, \bar{x}]$) it is possible to show that there exists *at least one stationary distribution* for the S.R.S in eq. (20), under weaker assumptions than for proposition 6. In particular, assumption 16 is not necessary and existence follows from Theorem 12.10 in Stokey and Lucas (1989). We are however unable to offer a proof of global stability for the S.R.S in eq. (20). ◊

8.3 *Non-iid shocks*

We now revisit the issue of i.i.d shocks in the S.R.S, eq. (17). Consider a scenario wherein the random recharge,R_t, depends upon the current stock, as is seen in the case of aquifers in which recharge depends upon lateral flows which are a function of current stock. Denote by $L(X)$ the lateral flow, with L increasing and (possibly) concave. We work with a slightly less general formulation of the problem, using the log normal distribution, letting only the mean of the recharge be a function of the stock.

Let $\eta \overset{i.i.d}{\sim} LN(\mu,\sigma)$, and consider the following random variable: $R_t = \eta + L(X_t)$, with $\mathbb{E}(R) = \mathbb{E}(\eta) + L(X)$ and $\mathbb{V}(R) = \mathbb{V}(\eta)$ (both conditional on X_t). R_t is a random variable with a so-called "shifted" log normal distribution, whose mean is increasing in X_t. Using this, we can reformulate our S.R.S as

$$\tilde{X}(X_t, w_t) = X_{t+1} = X_t + L(X_t) - w_t(X_t) + \eta_{t+1} \tag{19}$$

where η_t is now an i.i.d log normal random variable. To summarize, we have reduced an S.R.S with a random variable whose mean depends upon stock X_t into a slightly different S.R.S with an i.i.d recharge, with the addition of the term $L(X)$. Thus, no additional technical machinery is necessary to address the S.R.S in eq. (19). Nonetheless, the use of the new S.R.S, eq. (19), instead of the previous one, raises many questions, in particular:

1. Are the main results regarding the nature of the optimization problem, Theorems 1 and 2 still valid?[20]

2. Is the proof of global stability, proposition 6, valid?

Q1 can be answered in the affirmative, at least as regards theorem 1, since it only depends upon the monotonicity of the transition, $\tilde{X}$, in stock. We are unable to provide a proof that Theorem 2 and corollary 1 hold under this reformulation. Q2 can also be answered in the affirmative for the cost functions in eq. (6) and eq. (8), as we indicate below. We note, however, that the added complexity of this formulation necessitates correspondingly stronger assumptions; in other words, introduction of realistic features necessitate, in this case, slightly stronger assumptions. Nonetheless, we emphasize these assumptions are no stronger than those made in the existing literature, and we are unaware of any literature in economics which attempts to address explicitly this issue of dependence of the "shock" R upon the stock X, except for Stachurski (2003), with a different (multiplicative shock) formulation.

Letting $\tilde{a}(x)\,(= a(x) + L(x))$ be the new "reinvestment function", we note that the addition of $L(X)$ does not alter any of the previous properties regarding $w_t^*(x_t)$ in theorem 1 (and theorem 3 for the two cost functions in eq. (6) and eq. (8)), as already noted. Thus, whenever $a()$ is increasing in x, so is $\tilde{a}()$. However, in cases where $a()$ is not increasing in x, which is true for the cost functions in eq. (8) and eq. (6), no conclusive statement can be made regarding $\tilde{a}$ (for e.g., if L is sufficiently 'large', $\tilde{a}$ may be increasing even if a is not). For our purposes, we will *not* assume that $\tilde{a}$ is increasing in x whenever a is not.

Our main result for this part is stated below and note that proof of this result proceeds along lines identical to that of proposition 6.

Proposition 7. *theorem 4 is applicable, if $a(x)$is replaced with $\tilde{a}(x)$, for cost functions in eq.* (6) *and eq.* (8).

As for proposition 6, an analogue of assumption 16 is required; essentially, we replace a in assumption 16

[20]Note that Propositions 1-4 do not depend upon the S.R.S, except insofar as assumption 3 and assumption 11 are satisfied (which they are; the former since L is increasing, the latter a maintained assumption through out). theorem 3 follows if Theorems 1 and 2 hold under the assumed conditions.

with $\tilde{a}$ [21]. Since lateral flow may be viewed as being "small" (in relation to the stock, x), one can assume that $\gamma(x)$ is linear (or is dominated by a linear function) with slope η and, finally, that $\alpha > \eta$. Similarly, the condition that extraction be bounded away from 0, $\forall x > 0$, is evidently necessary here; in addition, given that there is a positive (in x) term, that condition is slightly more stringent[22]. Thus, the results for a are easily extended to accommodate $\tilde{a}$[23].

9 Conclusions and Extensions

The paper had two major objectives, to

i investigate the implications of using more realistic formulations of cost functions for dynamic groundwater management, including accounting for risk aversion and

ii provide weaker conditions for convergence of stock of resource.

The cost function was generalized in two directions, accounting for localized cones of depression (increasing the cost of extraction) and taking into account changes in groundwater stock within a season. With the conventional cost function, it was shown that only very few structural properties hold, notably monotonicity of extraction in groundwater stock. Quite surprisingly, this simple and intuitive result has been rigorously proved here for the first time. It was shown, with the former generalization of the cost function, that extraction is increasing in the current stock. With the latter generalization, however, it was shown, in addition, that reinvestment (next periods stock) was increasing in current groundwater stock and further, that extraction (reinvestment) was decreasing (increasing) over time.

In other words, it was shown that a longer horizon (very intuitively) leads to slower extraction and more reinvestment. We stress that our results appear to be the first to show these properties for any form of the dynamic groundwater extraction problem in the literature. Further, we also illustrate, with examples, that many of these results do not hold for the conventional (simplified) cost function.

In addition, we show that, when restricted to the log utility function, for both the conventional net benefit function as well as for the net benefit function accounting for cones of depression around the well, all of the preceding results are directly applicable. Again, apart from Knapp and Olson (1996), in whose set up uncer-

[21] This is a more stringent assumption than for the case without stock-dependent recharge, as is easily seen, since even for $x > x(1-\alpha) + \gamma(x)$, some form of lower bound on α is necessary.

[22] To see this, note that the condition now needed to be satisfied is: $w(x) > x(1-\alpha) + \gamma(x) = x(1-(\alpha-\eta))$, with $\alpha > \alpha - \eta > 0$ by assumption.

[23] Possible values for α are (i) $\alpha = 1 - \tilde{\beta}$, with $0 < \tilde{\beta} := \inf\left\{\frac{\partial w(x)}{\partial x} - \frac{\partial v(x)}{\partial x}; x > 0\right\} < 1$, a more demanding condition than for a (e.g., if w and v are dominated by linear functions with slopes η_w and η_γ, then it is necessary and sufficient that $\eta_w > \eta_\gamma$.) (ii) if $0 \leq \tilde{K} := \sup\{x + \gamma(x) - w(x); x > 0\} < 1$, then $\alpha = \tilde{K}$ (iii) if $0 < \tilde{\kappa} := \inf\left\{\frac{w(x) - \gamma(x)}{x}; x > 0\right\} < 1$, then $\alpha = 1 - \tilde{\kappa}$.

tainty plays a less central role in decision making, ours appears to be the only application explicitly proving structural results for objective functions displaying risk aversion. In contrast to most dynamic stochastic models in the resource and agricultural economics literature, we are able to prove most of the properties previously conjectured in the literature for these class of models.

Moving away from models which require strong assumptions such as monotonic reinvestment (two of our cost functions do not satisfy this condition) or compact state space, we illustrate the use of a powerful method for proving convergence of the stock of resource to a unique invariant distribution. This method requires only very mild assumptions on the optimal policy to yield convergence. The results obtained here are potentially applicable to a wide variety of renewable resource settings whose equation of motion is linear in stock and extraction.

The major objective of the paper was to modify existing groundwater management models to accommodate specific scenarios encountered in many semi-arid countries, scenarios previously not modeled (or at least, not with models specifically set up to accommodate features pertaining to these settings). This motivation suggests direct implications of the work here for informing public policies, although such extensions are left for future work. First and foremost, one or both cost functions introduced here may be used (as appropriate) to quantify the benefits of different types of real-world policies currently being considered (e.g. flat rate with capacity restricitons, following Gujarat's *jyotigram* example, as advocated in Shah et al. (2008)), with relatively minimal data requirements; such an evaluation with existing cost functions is likely to lead to substantial underestimates of the cost of pumping (and are possibly an important reason for the much debated and widely documented result regarding lack of benefits to regulation, see e.g. Brozović et al. (2010)). The hydrologically rigorous and realistic multi-user framework laid out in Brozović et al. (2006) requires detailed data on pump locations and more hydrological/economic parameters than are likely available for many developing countries, or at many locations[24]; in such cases, simulations using the models outlined here may provide a second best and relatively quick means of evaluating the benefits of different policies (such as taxes, considered in Athanassoglou et al. (2012)). Secondly, by freeing the "reinvestment function" from the artificial constraint of monotonicity, a more realistic baseline is provide for evaluation of policy (including using the conventional cost function). Essentially, policy simulations of the kind carried out in Knapp and Olson (1996, 1995) may be rendered more realistic and useful in the real world, using our approach, for the scenarios envisaged.

[24]To illustrate, in the single-user case, different policies may be relatively easily evaluated, and intuitively interpreted, with only data on average depth to water, some measure of porosity etc, following for e.g. Fishman et al. (2011). To extend the analysis in Athanassoglou et al. (2012), carried out for the case of two hypothetical users in the Telangana region in Southern India, to real-world aquifer systems with many users and with complex hydro-geology (for Telangana) is likely infeasible: indeed, in the absence of generalizability to other sites and contexts, such an extension may not, in the short-run, be very policy relevant at an aggregate spatial scale.

Two directions for extension of the current work are immediately evident. First, in the case of groundwater, an important policy issue is the prevention of groundwater depletion (or restoration of the groundwater system, if already depleted) by means of regulation. In this context, even if the optimal policy is monotone, it need not be a simple function of the stock. In the general setting of this paper, an important step towards specific applications could involve characterizing (even in the admittedly simplified case of a single user) a variety of possibly sub-optimal but simple or rule-of-thumb policies (e.g. a linear policy, as in Athanassoglou et al. (2012)) and quantifying their performance in an empirical setting.

Second, in the groundwater scenario which motivated this work, crop choice determines water use (and groundwater extraction) and exogenous prices determine crop choices. In this setting, it is important to understand the conditions under which the variance (variability) in prices influences the evolution of groundwater stock. In other words, the question of whether increases in variability of price leads to increase in extraction of stock is of some importance for designing public policy (e.g. price support and stabilization etc). Exploring this question in the model framework above, and extending the sparse existing literature on variability and stock exploitation (e.g. Sethi et al. (2005)), we feel, is both feasible and interesting.

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A Definitions and Notations

A.1 *Lattice Theory*

We enumerate below a list of assumptions, not all of which are used in all theorems or with all cost functions. We also set up some notation for future reference. Let $X \subset \mathbb{R}_+$be the state space (a lattice)[25], $W : X \to L(X)$, with $L(X)$ the set of all sub-lattices of X[26], $\mathscr{C} = \{(x,w); x \in X, w \in W(x)\}$ be a sub-lattice of $\mathbb{R}^2_+$.

Definition 4. (Ascending Set-Valued functions) The set-valued function W is called *ascending in x on X* (or simply "ascending") if it is *increasing on X*. In this definition, if x_1 and x_2 are in X, then $W(x_1)$ and $W(x_2)$ are in $L(X)$. Therefore, if $x_1 < x_2$, $a \in W(x_1)$ and $b \in W(x_2)$ and W is ascending on X, then necessarily $a \wedge b \in W(x_1)$ and $a \vee b \in W(x_2)$ [27].

Definition 5. (Expanding Sets)The set $W(x)$ is called "expanding" if $x' < x \implies W(x) \subset W(x')$.

Definition 6. (Stochastically supermodular) $\tilde{X}(x,w)$, a random variable parametrized by (x,w), defined on a lattice $X \times X$, $F_{x,w}$ its distribution function, is said to be "stochastically supermodular" in (x,w) if either of the following two conditions are satisfied:

1. (Topkis, 1998, pp 159) $\int_S dF_{x,w}(s)$ is supermodular in (x,w).
2. $\int h(s) dF_{x,w}(s)$ (as a deterministic function of (x,w)) is supermodular in (x,w), for all increasing and bounded functions h.

A.2 *Convergence of Markov Processes*

We continue with some notation, used below to elucidate ideas. Denote by $\mathscr{P}(X)$ the set of all (Borel-) probability measures on the state space X, a (Borel-) subset of $\mathbb{R}_+$ and denote by ψ any invariant distribution of the S.R.S in eq. (16). P is the *stochastic kernel* for the S.R.S in eq. (16), defined for $B \in \mathscr{B}(X)$ (set of Borel subsets of X) and $x \in X$, as

$$P(x,B) = \int \mathbb{I}_B \left[F(x,z)\right] \Phi(dz)$$

with $\mathbb{I}_B(.)$ an indicator function for the set B[28]. Given the stochastic kernel P, an associated linear operator,

[25] A partially ordered set X is a lattice if, $\forall x,y \in X$, it is the case that $x \wedge y$ and $x \vee y \in X$. When, as here, $X \subset \mathbb{R}$, with the usual order, then $x \wedge y = \min\{x,y\}$ and $x \vee y = \max\{x,y\}$.

[26] Intuitively, a set Γ is a sub-lattice of X if, $\forall x,y \in \Gamma$, it is the case that $x \wedge y$ and $x \vee y \in \Gamma$

[27] For ex. if $x \in X \subset \mathbb{R}$, $W(x) = [-\infty, x]$ or $W(x) = [x, \infty]$ are ascending on X.

[28] To understand the definition of the stochastic kernel, observe that $P(x_t, B) = \mathbb{P}(F(x_t, R_{t+1}) \in B) = \mathbb{E}\mathbb{I}_B\left[F(x_t, R_{t+1})\right]$. Two important properties of this kernel are: for every $x \in X$, $P(x,.)$ is a probability measure and for $B \in \mathscr{B}(X)$, $P(B,.)$ is a measurable function on X. Note finally that the definition of a stochastic kernel above is adapted to the setting of a S.R.S. The general definition of a stochastic kernel is as a family of probability measures. Notationally, a general stochastic kernel is represented as $P(x,dy) \in \mathscr{P}(X)$, $(x \in X)$.

the Markov Operator $M : \mathscr{P}(X) \to \mathscr{P}(X)$ maybe defined as

$$\Phi M(B) := \int P(x,B)\Phi(dx).$$

The Markov operator is therefore a linear operator mapping distribution functions from $\mathscr{P}(X)$ to $\mathscr{P}(X)$. Let $X_0 \sim \Psi$, $\Psi \in \mathscr{P}(X)$, and denote the distribution of $(X_t)_{t\geq 0}$ as Ψ_t. Then it is the case that the recursion $\Psi_{t+1} = \Psi_t M$ holds, which yields, by an inductive argument, the infinite-dimensional version of the usual Markov Chain identity: $\Psi_{t+1} = \Psi M^t$.

The operator M also acts on *functions* and is used to define the expectation of any function w.r.t the stochastic kernel i.e. $Mh(x) := \int h(y)P(x,dy)\ x \in X$. In the context of a S.R.S., this is equivalent to the operationally useful definition: $Mh(x) := \int h[F(x,z)]\Phi(dz), x \in X$. A final point regarding the operator M is that it acts on distributions (alternatively measures) to the *left* and functions to the *right*. We now collect a few formal definitions of the properties of P and M, indicating also sufficient conditions for these properties to be fulfilled. We list these properties of M and P since they are integral in the proofs of both our main theorems. More details on these properties, including those regarding the recursions stated above and the linearity of M, may be found in many standard texts on Markov processes (cf. Stachurski (2009)(Chapters 8, 9.2 and 11), Stokey and Lucas (1989)(Chapter 12) and Meyn and Tweedie (1993)).

Definition 7. (Feller Property) A Markov operator M is said to possess the "Feller Property" if it maps bounded, continuous functions into bounded continuous functions.

Remark 14. (Sufficient condition for Feller Property) It is straightforward to prove that if the map $F(x,R)$ is continuous in x on X for each R, then a Markov operator satisfies the Feller Property (see definition 7 in Appendix). In fact, for F linear (affine) in x, as in the S.R.S in eq. (17), with $F(x,R) = a(x)+R$, it is sufficient that F is bounded (since continuity is immediate).

Considering instead the S.R.S corresponding to the transition equation for a finite aquifer in eq. (2),

$$X_{t+1} = F(x_t, R_{t+1}) = \min(a(x_t) + R_{t+1}, \bar{x}) \tag{20}$$

it is evident that due to the continuity of $\min(a(x), \bar{x})$ in x, $F(x,R)$ is again continuous and therefore, satisfies the Feller property. Finally, by similar reasoning, it is evident that the S.R.S in eq. (19) is also continuous in x and therefore, satisfies the Feller property. $\Diamond$

Definition 8. (Iterates of M) We have already indicated that $Mh(x)$ may be interpreted as the expectation of the function h under the kernel P. We now illustrate the connection between the t^{th} iterate of M applied to h and the same iterate applied to a distribution Ψ. Note first that since $P^t(x,dy)$ may be interpreted as the distribution of X_t given $X_0 = x$, it is immediate that $M^t h(x)$, defined as

$$M^t h(x) := \int h(y) P^t(x,dy) = \mathbb{E}\left(h(X_t)|X_0 = x\right) \ (x \in X)$$

may be interpreted as a conditional expectation. Then, it can be shown (Stachurski (2009) §9.2 (Thm 9.2.15)) that

(a) $\Phi(Mh) = (\Phi M)(h) = \int \left[\int h(y)P(x,dy)\right] \Phi(dx)$

(b) By induction, $\Phi(M^t h) = (\Phi M^t)(h) = \int \left[\int h(y)P^t(x,dy)\right] \Phi(dx)$

Thus, $\Phi(M^t h)$ is merely an (unconditional) expectation.

Definition 9. (Aperiodic Kernel) A kernel P is said to be *aperiodic* if it has a $(\nu,\varepsilon)-$small set C with $\nu(C) > 0$.

Definition 10. (Irreducible kernel) A kernel P is said to be $\mu-$irreducible, with $\mu \in \mathscr{P}(X)$, if $\forall x \in X$ & $B \in \mathscr{B}(X)$ with $\mu(B) > 0$, $\exists t \in \mathbb{N}$ s.t. $P^t(x,B) > 0$. If P is irreducible for an arbitrary $\mu \in \mathscr{P}(X)$, then it is called *irreducible*.

Definitions 9 and 10 are simply infinite state analogues of the classical definitions for finite state Markov Chains.

Definition 11. (Stability) Let Ψ^*be an invariant distribution. Denoting by $ib(X)$ $(ibc(X))$ the set of increasing and bounded (continuous) functions on X, *stability* of Ψ^* is taken to mean

$$\forall \Psi \in \mathscr{P}(X) \text{ \& } h \in ib(X), \ \left(\Psi M^t\right)(h) \to \Psi^*(h) \text{ as } t \to \infty \tag{21}$$

This non-standard condition may be easily understood as a stronger form of convergence than weak convergence of the (measure) distribution Ψ. To see this, note that we have already defined $(\Psi M^t)(h)$ as an expectation, in definition 8. Therefore, weak convergence of Ψ would imply the convergence in eq. (21) for $h \in \ ibc(X)$. However, noting that $ibc(X) \subset ib(X)$, it is immediate that the convergence in eq. (21), which holds for $h \in ib(X)$, is stronger than weak convergence.

Definition 12. (Small set) Let $\nu \in \mathscr{P}(X)$, $\varepsilon > 0$. A set $C \subset B(X)$ is called $(\nu,\varepsilon)-$small for P if $\forall x \in C$, it is the case that for $A \in \mathscr{B}(X)$, $P(x,A) \geq \varepsilon\nu(A)$. If this condition holds for *some* ν and $\varepsilon > 0$, then the set C is called *small*.

B Proofs

Proof of theorem 1

Proof. That $V_t(x)$ is increasing in x is a straightforward consequence of two facts, $V_{T+1}(x) = 0$ and $\Pi(x, w)$ is increasing in x, given w. These both facts can be used to set up a simple inductive argument, as below.

$V_{T+1}(x)$ is trivially increasing in x, and so the induction is true for $t = T+1$. Similarly, $V_T = \max\{\Pi(x, w); w \in W(x)\}$ is increasing in x, since Π is increasing and $W(x)$ is ascending and expanding and so the hypothesis is true for $t = T$. Let it be true for $t = k < T$ i.e. $V_k(x)$ is increasing in x. Consider $V_{k-1}(x) = \max_{w \geq 0}\{J_{k-1}(x, w)\}$, where $J_{k-1}(x, w)$ $(= \Pi(x, w) + \delta\mathbb{E}(V_k(x - w + R)))$ is increasing in x, since both its terms are (the second term, $\mathbb{E}(V_k(x - w + R))$, is increasing by the induction assumption and the fact that integration is order preserving) and thus, so too is V_{k-1}. As an aside, observe that if Π is bounded, so is V_t.

We next prove that $\mathbb{E}\left[V_t\left(\tilde{X}(x_t, w_t)\right)\right]$ is supermodular in (x, w). We have already proved that V_t is increasing and bounded, and have assumed that the distribution function of $\tilde{X}(x_t, w_t)$, $F_{x,w}$, is supermodular. Therefore, by Topkis (1998)[Corollary 9.1(b), pp 160], $\mathbb{E}\left[V_t\left(\tilde{X}(x_t, w_t)\right)\right]$ is supermodular. Thus, we have that $J_t(x_t, w_t)$ in eq. (9b) is supermodular (since the sum of two supermodular functions is supermodular).

Since $W(x)$ is ascending and expanding in x, that $w_t^*(x) = argmax\{J_t(x_t, w_t)\}$ is non-empty and ascending is the content of Topkis (1998)[Theorem 2.8.1, pp 78]. Finally, if $W(x)$ is compact, then from Topkis (1998)[Theorem 2.8.3(a), pp 78], $w_t^*(x)$ is a sub-lattice of $\mathbb{R}_+$, with a least and a greatest element, both of which are increasing in x. □

Proof of theorem 3

Proof. We split the proof outline into two steps. The first step involves proving that (a) $V_t \downarrow V$ and that (b) V satisfies eq. (15a). (a) is remark 8. A proof of (b) begins by noting that since each V_t is bounded and increasing in x, by the monotone convergence theorem for integrals, we have that

$$\lim_{t\to\infty} \mathbb{E}\{V_t[\tilde{X}(x, w)]\} = \mathbb{E}\left\{V\left[\tilde{X}(x, w)\right]\right\}. \tag{22}$$

From this, it is immediate that

$$V_t(x) = \max\{J_t(x, w)\} \geq \max\{J(x, w)\}$$

which implies

$$\lim_{t\to\infty} V_t(x) = V(x) \geq \max\{J(x, w)\}$$

The proof will be complete if the inequality is proved the other way i.e. if it is established that $V(x) \leq \max\{J(x,w)\}$. To prove this, consider

$$V(x) \leq V_t(x) = \sup\{J_t(x,w); w \in W(x)\}$$

and taking limits, observe that

$$V(x) \leq \lim_{t\to\infty} [\sup\{J_t(x,w); w \in W(x)\}]$$

The proof is complete if

$$\lim_{t\to\infty} [\sup\{J_t(x,w); w \in W(x)\}] = \sup\left[\lim_{t\to\infty} \{J_t(x,w); w \in W(x)\}\right]$$

i.e. if the operation lim and sup can be interchanged. From Assumptions 14 and 10, it follows that $J_t(x,.)$ converges uniformly in w, for each x, to $J(x,.)$ and finally, eq. (22) justifies interchange of integral and limit. Thus, the interchange of lim and sup is valid.

The next step, to prove that $w^*(x)$ is ascending in x and the greatest element of $w^*(x)$ is increasing in x, is a direct consequence of theorem 1, and is plainly a result of the supermodularity of J. $\square$